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Evolution of matter density perturbations in viable $f(R)$ theories of gravity

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"From the End of the Rainbow to the Edge of Time, A Journey Through the Wonders of Physics"
Walter Lewin

ABSTRACT

In the Λ CDM model, the late-time accelerated expansion of the Universe is explained via a dark energy fluid in the form of a cosmological constant. Such a cosmological constant dominates the energy budget of the Universe today, and yet, it is still a poorly understood species because it is not observed yet. A competitive theoretical approach to understand this is via the so-called $f(R)$ extended theories of gravity, which explain the late acceleration epoch of the Universe resorting to a geometrical modification of the field equations. We illustrate how $f(R)$ theories are constructed and how both the analysis of the cosmological expansion and the growth of matter density perturbations in these theories may differ from the standard Einsteinian results. We study the evolution of matter density perturbations in a viable $f(R)$ model (Hu-Sawicki model) and explain why the Hu-Sawicki model is indeed a viable alternative to Λ CDM by discussing the Dynamical System approach as a method used to obtain the cosmological background solutions. A complete comparison of density perturbations in both the Λ CDM model and Hu-Sawicki model is presented.

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AUTHOR'S DECLARATION

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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0.1 Notation/Conventions

The following notations and conventions are used in this dissertation. The metric signature convention used is $(-+++)$ and the Minkowski metric is denoted by $\eta_{\mu\nu}$. In 4-dimensional space, the metric takes the form $\text{diag}(-1,1,1,1)$ in the Cartesian coordinates. where (μ, ν) run over four dimensions of spacetime, i.e., $\nu, \mu = 0, 1, 2, 3$ and (i, j) run over three spatial dimensions and η represents conformal time. We also adopted the natural units, the Newton's constant G , the speed of light c and the Planck's constant \hbar , i.e., $(c = \hbar = 1)$.

The prime denotes the derivative with respect to the conformal time, $f(R)$ is the function of the Ricci scalar R and f_R represents the first derivative of $f(R)$ with respect to the Ricci scalar, Similarly the f_{RR} represents the second derivative of the function $f(R)$, i.e., $' \equiv d/d\eta$, $f \equiv f(R)$, $f_R \equiv df/dR$ and $f_{RR} \equiv d^2f/dR^2$. The square term is defined as $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$, where $g^{\mu\nu}$ is the inverse of the metric $g_{\mu\nu}$, the $;$ represents the derivative in the Equations and the ∇ is the covariant derivative with respect to the unperturbed Friedmann-Lemaître-Robertson-Walker metric.

INTRODUCTION

1.1 Introduction

The study of modern cosmology is seeking the fundamentals to understand the nature of the fluids and geometry of the Universe. This task involves the development of theoretical ideas about the nature of gravity, to be compared with the observations that cosmic probes have been undertaking. The current knowledge and understanding of cosmology are not sufficient to answer some of the biggest and interesting questions about our universe. General Relativity (theory of gravity) has answered most of these questions precisely, but it does not explain everything in cosmology and astrophysics domain [1].

The concept of dark matter and dark energy is currently dominating in cosmology and poses some questions about our existing theory of gravity, this is one of the reasons that led to an idea of extending the existing theory of gravity in order to accommodate the outliers (i.e., the late-time cosmic acceleration of the Universe). Present understanding of cosmology is condensed in the standard model that contains the material content of the standard model of particle physics and Einstein's theory of General Relativity with a cosmological constant.

The predictions have been made, tested and confirmed, although there are still some issues that remain open. One of these unquestionable issues is the rise or existence of the unknown energy (dark energy) and dark matter in the Universe, not only that but even their nature and detailed properties are still not clear. Nevertheless, the studies have confirmed some specific and important roles that these dark components play in Astrophysics and Cosmology.

It has been confirmed by cosmic measurements, that their influence is at least gravitationally and this knowledge allows us to view the current gravity theory in a different perspective (extension of General Relativity) hence in this present work we examine the theoretical ideas to extend the known Einstein's General Relativity so that we can be able to study beyond GR itself. Thus this

study is investigating the alternative way to understand the nature and origins of dark energy without postulating the cosmological constant. Modified theories of gravity have been developed in an attempt to answer this question of "dark energy fluid". Among these, $f(R)$ theories of gravity have attracted many scientists, it is more competitive following its logical construction from General Relativity which is still the best theory to explain gravity. Its gravitational action contains a generic function of the Ricci scalar hence called $f(R)$. Similarly, a variety of $f(R)$ models have been proposed with one main goal of explaining the effect of dark energy fluid without using the concept of the cosmological constant. However none of them is perfect enough, $f(R)$ models are proposed that they should satisfy the so-called standard viability criteria in order to be successful both theoretically and observationally [1]. Many $f(R)$ models are facing the question of how to discriminate between them by using present and future observations, because the FLRW metric can be taken as a solution for most gravitational field equations. However, the evolution of perturbations is sensitive not only to the background evolution but also to the adopted theories of gravity, which means that different theories of gravity produce different cosmological perturbations which leave different relics in the universe [1].

1.1.1 General Relativity

A quick recap of General Relativity (GR) will be necessary for this study, since it will outline the important concepts that are required to understand cosmology. Therefore I found it useful to review GR first before getting into cosmology itself. The Einstein theory (GR) is based on the study of differential geometry and it describes the gravity of curved spacetime based on two fundamental principles:

- (i) **The principle of equivalence:** *free-falling observers within a gravitational field are locally equivalent to inertial observers.*
- (ii) **The principle of general covariance:** *the laws of physics must have the same form in all frames of reference*

In GR, matter and energy are equivalent. This means that the space-time curvature is related to the stress-energy tensor $T_{\mu\nu}$, which contains all the information about matter and energy of a system [2]. The interaction between spacetime and matter is governed by:

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \quad (1.1)$$

where $G_{\mu\nu}$ can be written in terms of the Ricci tensor and the Ricci scalar as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \quad (1.2)$$

the term $T_{\mu\nu}$ is the stress-energy tensor and $G_{\mu\nu}$ is a purely geometric tensor containing the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R . Lastly κ is a constant of proportionality which contains the Newtonian gravitational constant G (i.e., $\kappa = 8\pi G/c^4$)

1.1.2 Perturbative General Relativity

A breakthrough was made when Albert Einstein realized that gravity can be geometrically interpreted. Einstein concluded that, what we experience as gravity is due to the intrinsic curvature of spacetime. Spacetime is defined as a manifold whose points correspond to physical events which are represented by four coordinates written as a four-vector, $x^\mu = \{x^0, x^1, x^2, x^3\}$ [3], normally in Cartesian coordinates this is chosen as $\{t, x, y, z\}$.

In the perturbation theory of GR, we consider two different spacetimes, one is the perturbed spacetime and the other is the background spacetime. Therefore in this work, we will keep referring to these two different spacetimes, see Fig. (1.1)

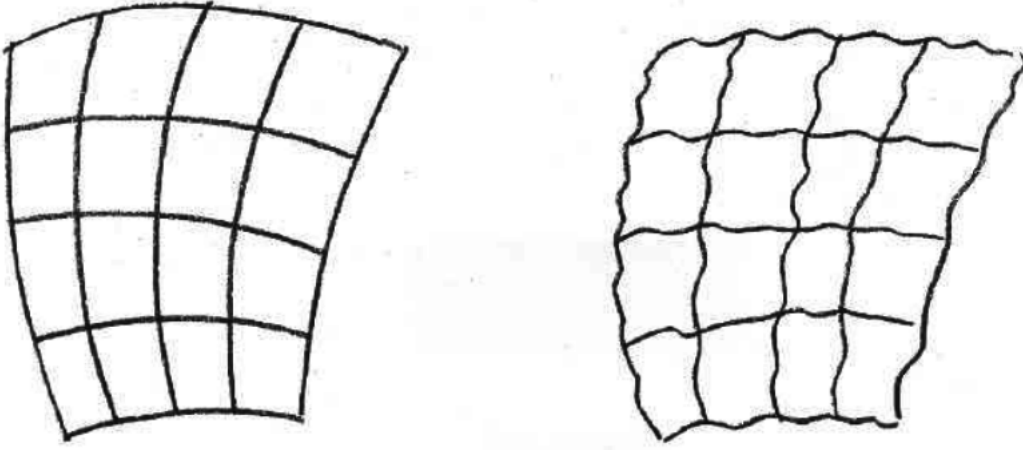


Figure 1.1: The background spacetime and perturbed spacetime [4].

This simply means that there exists a coordinate system on the perturbed spacetime, whose metric is given by

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad (1.3)$$

where $\bar{g}_{\mu\nu}$ is the metric of the background spacetime and $\delta g_{\mu\nu}$ is considered to be very small, meaning the first and second $(\delta g_{\mu\nu,\rho}, \delta g_{\mu\nu,\rho q})$ partial derivatives will be even much smaller, such that they can be neglected.¹ Perturbation analysis provides with an essential supporting structure for understanding the effects of an object of a small mass " m " moving through a "background" spacetime, hence in the perturbed spacetime the curvature and energy tensors can be written as follows:

$$\begin{aligned} G^\mu_\nu &= \bar{G}^\mu_\nu + \delta G^\mu_\nu, \\ T^\mu_\nu &= \bar{T}^\mu_\nu + \delta T^\mu_\nu, \end{aligned} \quad (1.4)$$

¹Let us continue to refer the background quantities with the overbar.

where the quantities δG_ν^μ and δT_ν^μ are considered to be very small.

We then require a point-wise correspondence between the two spacetimes, so that we can perform the comparisons and subtractions, which comes from the coordinate system $\{x^0, x^1, x^2, x^3\}$. Note the two different points from different spacetimes, i.e., point \bar{P} from the background spacetime and point P from the perturbed spacetime have the same coordinate system.

Then the subtraction of the two Einstein's equations from two different spacetimes gives

$$G_\nu^\mu = 8\pi G T_\nu^\mu \quad \text{and} \quad \bar{G}_\nu^\mu = 8\pi G \bar{T}_\nu^\mu, \quad (1.5)$$

and hence

$$\begin{aligned} G_\nu^\mu - \bar{G}_\nu^\mu &= 8\pi G T_\nu^\mu - 8\pi G \bar{T}_\nu^\mu, \\ &= 8\pi G [T_\nu^\mu - \bar{T}_\nu^\mu], \\ &= 8\pi G [\bar{T}_\nu^\mu + \delta T_\nu^\mu - \bar{T}_\nu^\mu]. \end{aligned} \quad (1.6)$$

But we know that

$$G_\nu^\mu - \bar{G}_\nu^\mu = \delta G_\nu^\mu. \quad (1.7)$$

Therefore Eq. (1.6) can be written as

$$\delta G_\nu^\mu = 8\pi G \delta T_\nu^\mu, \quad (1.8)$$

which are the field equations for the perturbations.

In general, given a coordinate system on the background spacetime, there exist a variate number of coordinate systems for the perturbed spacetime, for which Eq. (1.3) holds.

Later on, we will discuss the so-called "gauge choice", which is basically the choice of these coordinate systems [4].

In this thesis, we studied the *cosmological perturbation theory* by considering the background spacetime to be the Friedmann-Lemaître-Robertson-Walker Universe. Therefore background spacetime is curved and not empty, isotropic and homogeneously time-dependent.

We then obtain two types of perturbations, the first-order type as well as the second-order type.

In **first-order perturbation theory**: this type of perturbation, it results from dropping all the terms that contain products of small quantities $\delta g_{\mu\nu}$, $\delta g_{\mu\nu,\rho}$ and $\delta g_{\mu\nu,\rho q}$, which then makes the Eq. (1.8) to be linear differential equation for $\delta g_{\mu\nu}$. The simplest case is where the background is the Minkowski space. then $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\bar{G}_\nu^\mu = \bar{T}_\nu^\mu = 0$.

In **second-order perturbation theory**: here we keep those terms with a product of two (but not more) small quantities.

1.1.3 The Gravitational Action

In the previous section, I briefly discussed the necessity of GR as a tool towards understanding the late-time Cosmology. In fact, it is important to learn the fundamentals of GR, how it is constructed to explain the space-time and gravity. Gravity is well explained by GR, which is described by Hilbert-Einstein Action that leads to Einstein's Field Equations through the principle of least action.

The Hilbert-Einstein Action is given as [5]:

$$S_{GR} = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa} + \mathcal{L}_m \right], \quad (1.9)$$

where \mathcal{L}_m is the Lagrangian of matter and $g = \det(g_{\mu\nu})$ is a determinant of the metric tensor matrix. The Einstein's Field Equations Eq. (1.1) are found by varying Eq. (1.9) with respect to the metric.

1.2 Field Equations

1.2.1 Einstein Field Equations

It was previously mentioned how the Einstein's Field Equations are obtained from Eq. (1.9). In this section, we show in details how this was achieved.

We start by considering our Action, we then recall the principle of least action, which states that the variation of Eq.(1.9) with respect to inverse metric is equal to zero, i.e., $0 = \delta S$.

Then after using Eq.(1.9) it follows as,

$$\begin{aligned} 0 &= \delta S, \\ &= \int \left[\frac{1}{2\kappa} \frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} d^4x, \\ &= \int \left[\frac{1}{2\kappa} \left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \sqrt{-g} d^4x. \end{aligned} \quad (1.10)$$

Since this equation should hold for any variation $\delta g^{\mu\nu}$, it implies that

$$\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = \frac{-2\kappa}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \quad (1.11)$$

The right hand side of Eq.(1.11) is directly proportional to the stress-energy tensor [6, 7],

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} = -2 \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_m. \quad (1.12)$$

To calculate the left hand side of the Eq. (1.11), we need the variations of the Ricci scalar R and the determinant g of the metric $g_{\mu\nu}$; these can be easily calculated, please refer to [6] for deeper understanding. To calculate the variation of the Ricci scalar we calculate first the variation of the Riemann curvature tensor, and then the variation of the Ricci tensor. So, the Riemann curvature tensor is defined as [8]

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \quad (1.13)$$

where Γ 's are the components of the Christoffel symbols and are given by:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (g_{\beta\nu,\alpha} + g_{\alpha\nu,\beta} - g_{\alpha\beta,\nu})$$

Now we recall that the Riemann curvature depends only on the Levi-Civita connection, so the variation of the Riemann tensor can be as follows:

$$\delta R^\rho{}_{\sigma\mu\nu} = \partial_\mu \delta \Gamma^\rho_{\nu\sigma} - \partial_\nu \delta \Gamma^\rho_{\mu\sigma} + \delta \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\mu\lambda} \delta \Gamma^\lambda_{\nu\sigma} - \delta \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\nu\lambda} \delta \Gamma^\lambda_{\mu\sigma}. \quad (1.14)$$

Since $\delta \Gamma^\rho_{\nu\sigma}$ is the difference of two connections, it is a tensor and we can thus calculate its covariant derivative,

$$\nabla_\mu (\delta \Gamma^\rho_{\nu\sigma}) = \partial_\mu (\delta \Gamma^\rho_{\nu\sigma}) + \Gamma^\rho_{\mu\lambda} \delta \Gamma^\lambda_{\nu\sigma} - \Gamma^\lambda_{\mu\nu} \delta \Gamma^\rho_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma} \delta \Gamma^\rho_{\nu\lambda}. \quad (1.15)$$

By contracting two indices of the variation of the Riemann tensor, we obtain the variation of the Ricci curvature tensor and the Palatini identity [8]:

$$\delta R_{\sigma\nu} \equiv \delta R^\rho{}_{\sigma\rho\nu} = \nabla_\rho (\delta \Gamma^\rho_{\nu\sigma}) - \nabla_\nu (\delta \Gamma^\rho_{\rho\sigma}), \quad (1.16)$$

The Ricci scalar is defined as $R = g^{\sigma\nu} R_{\sigma\nu}$, therefore its variation with respect to the inverse metric $g^{\sigma\nu}$ is given by

$$\begin{aligned} \delta R &= R_{\sigma\nu} \delta g^{\sigma\nu} + g^{\sigma\nu} \delta R_{\sigma\nu} \\ &= R_{\sigma\nu} \delta g^{\sigma\nu} + \nabla_\rho (g^{\sigma\nu} \delta \Gamma^\rho_{\nu\sigma} - g^{\sigma\rho} \delta \Gamma^\mu_{\mu\sigma}). \end{aligned} \quad (1.17)$$

Note: we used the metric compatibility of the covariant derivative, $\nabla_\sigma g^{\mu\nu} = 0$ to achieve the above.

We can now re-name the dummy indices and multiply the last term of the second line in Eq. (1.17) by $\sqrt{-g}$ and we get the total derivative and applying Stokes's theorem we obtain,

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}. \quad (1.18)$$

Now we want a variation of the determinant, to do so we use the rule for differentiating a determinant, which gives us the following:

$$\delta g = \delta \det(g_{\mu\nu}) = g g^{\mu\nu} \delta g_{\mu\nu}.$$

One could transform into a coordinate system where $g_{\mu\nu}$ is diagonal to get

$$\begin{aligned} \delta \sqrt{-g} &= -\frac{1}{2\sqrt{-g}} \delta g, \\ &= \frac{1}{2} \sqrt{-g} (g^{\mu\nu} \delta g_{\mu\nu}), \\ &= -\frac{1}{2} \sqrt{-g} (g_{\mu\nu} \delta g^{\mu\nu}). \end{aligned} \quad (1.19)$$

It is not hard to follow the below conclusion

$$\frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu}. \quad (1.20)$$

Finally, we arrived at this stage, we have all the material to set in our equation of motion for the metric field to obtain,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (1.21)$$

and are the Einstein's field equation.

where

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu}, \quad (1.22)$$

and therefore Eq. (1.21) can be written as,

$$G_{\mu\nu} = \kappa T_{\mu\nu}. \quad (1.23)$$

It turned out that Eq. (1.23) failed to explain the so-called "static Universe " since the Universe was thought to be static. This is when Einstein himself decided to add the Lambda (Λ) term in his equations by including it in Eq. (1.9), i.e.,

$$S_{(GR+\Lambda)} = \int d^4x \sqrt{-g} \left[\frac{(R - 2\Lambda)}{2\kappa} + \mathcal{L}_m \right], \quad (1.24)$$

where $G_{\mu\nu}$ is a Einstein's tensor and Λ is "cosmological constant".

Which leads to "Cosmological Constant problem" defined as the disagreement between the observed values of vacuum energy density and theoretical value of zero-point energy suggested by quantum field theory. The Einstein's field equations for this case are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \quad (1.25)$$

1.2.2 Dark Energy Problem

As mentioned in the previous section, the cosmological constant describes the unknown energy density of the Universe called "dark energy" and has the same effects as an intrinsic energy density of the vacuum. The observed value of $\Lambda = 1.1056 \times 10^{-52} \text{m}^{-2}$, provided by Planck (2018) data [9].

Studies have shown that dark energy has the implication of negative pressure which results from positive vacuum energy. If the energy density is positive, the associated negative pressure will drive an accelerated expansion of the Universe as observed, see Fig. (1.2).

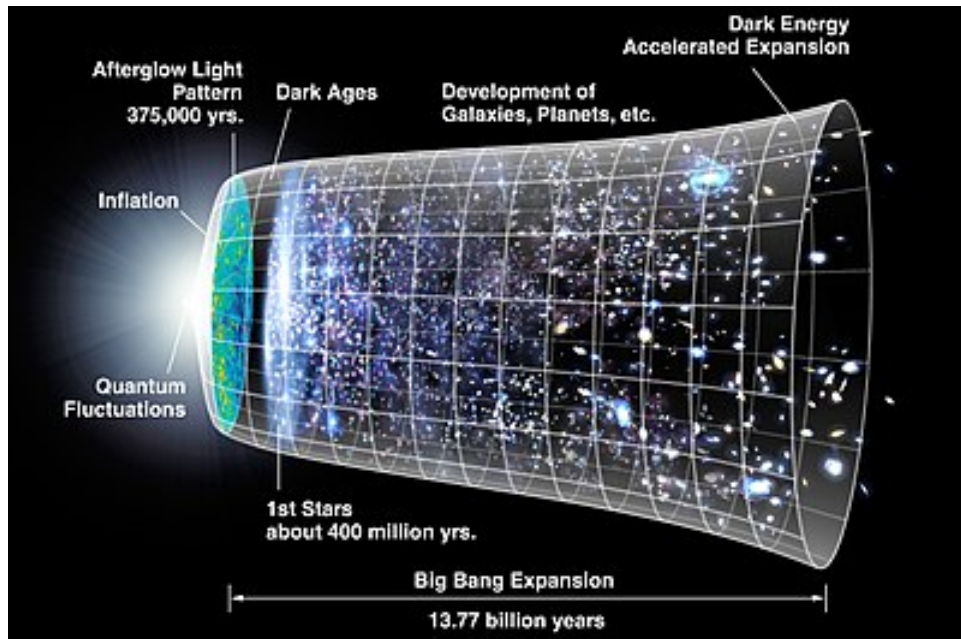


Figure 1.2: The timeline of the Universe in the Λ CDM model. The accelerated expansion in the last third of the timeline represents the dark-energy dominated era [10].

1.2.3 The Cosmological Coincidence Problem

The idea that our Universe is currently expanding at the accelerating rate is one of the fundamentals of this known cosmological problem (**Cosmological Coincidence problem**).

The observational evidence that the presently observed values of dark energy and dark matter densities have the same order of magnitude seems to indicate a special period in our universe [11].

The big question towards this conjecture is, "Why now?" which constitutes the cosmological coincidence problem that we know.

This surprising observation of these densities is based on the standard model of cosmology, i.e., the Universe is expanding at an increasing rate at the current time. Meaning if this expansion at an increasing rate is not true, that will simply imply that there is no "cosmological coincidence problem" [12].

The second question is "Why at an accelerating rate?", in Λ CDM model the acceleration rate of expansion of the Universe is explained by introducing the dark energy fluid, which is explained by the interpretation of redshifts of the observed behaviour of the galaxies in the universe.

Observations show that these densities of (dark energy and dark matter) are roughly equal at the small values of redshift, i.e., $z = 0.55$ [11] and this equality at this point suggest dark energy dominated era on the cosmological scale, giving rise to the question of "Why now". Until this far, based on standard model this question seems to be lacking a strong and a reasonable answer simply because there is an evidence that the expansion of space is accelerating and that results in introducing dark energy fluid which is assumed to be an energy density of space itself and until now nobody understands what this dark energy is, even though its anti-gravity effect is well understood.

This anti-gravity effect has been relatively small compared to the gravity effect of matter (normal and dark) over time in the history of the Universe. Since the Universe has been expanding, the matter density has been diluted to a level where the gravity effect of matter has only relatively recently been "overtaken" as it were by the anti-gravity effect of dark energy [13].

With all the problems facing the Λ CDM model, it is logically motivated to think about alternative ways of explaining the accelerating rate of expansion without using the concept of cosmological constant, hence the existence of extended theories of gravity.

1.2.4 Summary for Chapter 1

This chapter reviewed some interesting questions about the Universe. The first one is, why cosmology in the first place, the problems that cosmology is facing and how they arise. A short review on the GR was presented, its fundamentals and how it was constructed based on the study of differential geometry. An introduction on perturbations in GR was also studied in order to understand some basic ideas of cosmological perturbation theory which will be discussed later in the study. Finally, a detailed derivation of the field equations from the gravitational action (Hilbert-Einstein

Action) was presented. We also mentioned how and why the Λ term was introduced in the Einstein field equations, important because it is the fundamental of some big and challenging problems in cosmology like dark energy problem, coincidence problem and others which are briefly discussed as well.

CONCORDANCE COSMOLOGICAL (Λ CDM) MODEL

2.1 Modern Cosmology

2.1.1 Background model

According to observations, Our Universe is known to be homogeneous and isotropic on large scales. Observations reflect that the density in our observable Universe is the same everywhere (Homogeneity) and it looks the same in every direction (Isotropy). The statement of homogeneity and Isotropy is called **Cosmological Principle** [14]. This principle expresses that our position in the Universe is not special, meaning that an observer in any other galaxy can observe the same as we do. This is based on the fact that the temperature of the cosmic microwave background (CMB) photons have the same temperature and it is constant everywhere with the value of 2.73K. Observations provide with the information that our Universe is expanding in an accelerating rate and one of the reasons of this expansion might be the energy density of the Universe is dominated by dark energy (a theoretical repulsive force that counteracts gravity) which is represented by Λ in the Einstein field equations. The Λ term is believed to be responsible for accelerated expansion of the Universe. The Λ CDM is the current cosmological model which describes the cosmological constant and its geometries are characterized by a scale factor $a(t)$ that shows how relative spatial distances between fundamental world lines change as time progresses [15], for illustration, see Fig. (1.2).

2.1.2 Cosmological Principle Problem

Even though it cannot be mathematically proven, the numerous observations do support the concept of cosmological principle. The Universe is known to have no special place, hence the isotropy and homogeneity [5]. see Fig (2.1)

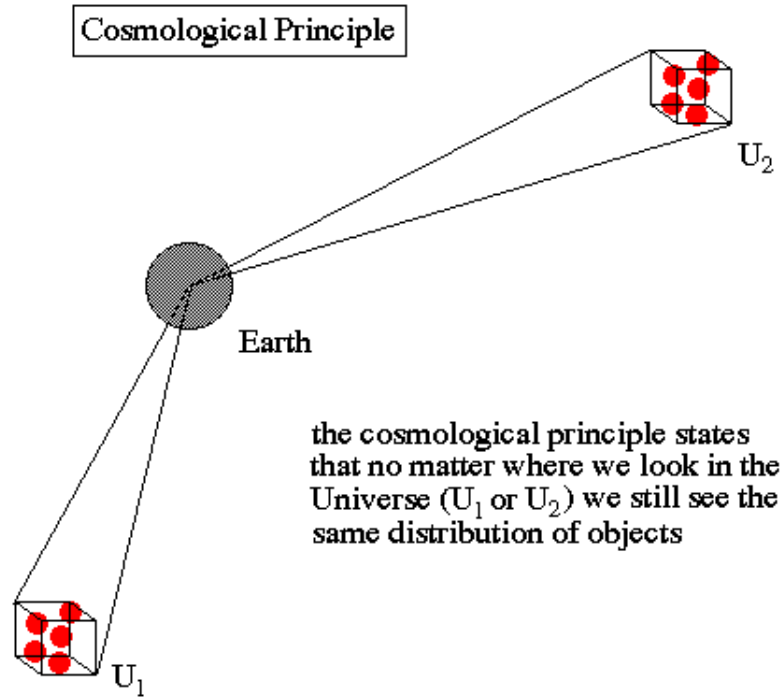


Figure 2.1: Two observers U_1 and U_1 observe the same Universe from different positions [16]

This idea results in a corollary that laws of physics should work the same everywhere in the Universe and the universal physical constants such as the speed of light, gravitational constant, mass of the electron are not changing as well in the Universe.

One clear evidence supporting cosmological principle is the measurements of CMB, this will be discussed in (Section 2.2.3). Homogeneity and Isotropy implies that obverations which were conducted in the past can be assumed to work under the same physics even today.

2.2 Observational expansion of the Universe

2.2.1 The history of expansion

On large scales (> 100 Mpc) the Universe is homogeneous and isotropic. This is known as the cosmological principle which is observationally supported by the fact that photons coming from every part of the sky nearly have the same temperature. These photons are what we refer as the CMB.

The current Standard Model of cosmology contains the (Λ term), which is known as the "dark energy fluid", which is responsible for the late-time accelerated expansion of the Universe. In

Friedmann Universe where the metric is given by Eq. (3.2), we have two important solutions to Einstein's equations that describe the rate of expansion of the Universe known as the Friedmann equations. Here, we are just giving the expressions for the two equations. More details in Chapter 3, Section (3.1.1).

The first equation is given by

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}, \quad (2.1)$$

where $H = \dot{a}/a$ is the Hubble rate and \dot{a} represents the rate of change of the scale factor a with respect to cosmic time.

The scale factor accounts for the expansion of the Universe and the total energy density of the Universe is given by ρ . G and k respectively represent, the Newtonian gravitational constant and the curvature parameter of the Universe. There is a second Friedmann equation known as the acceleration equation is [6],

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \quad (2.2)$$

where p is the pressure of the Universe.

The Universe is known to be composed of matter (baryon + cold dark matter), radiation and dark energy each of which follows the equation of state (EoS),

$$p = w\rho \quad (2.3)$$

which relates the pressure to the energy density via the EoS parameter w . If we replace Eq. (2.3) in Eq. (2.2) we have

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1 + 3w)\rho. \quad (2.4)$$

For an accelerating Universe we require $\ddot{a}/a > 0$ and this implies,

$$w < -\frac{1}{3}. \quad (2.5)$$

Hence, we see that dark energy is a fluid with negative pressure.

For the Λ CDM model we have $w = -1$ for dark energy (in the form of a cosmological constant) dominated Universe, $w = 0$ for matter dominated Universe and $w = 1/3$ for radiation dominated Universe [6], Below are different stages of the Universe as it evolved with time, see Fig.(2.2).

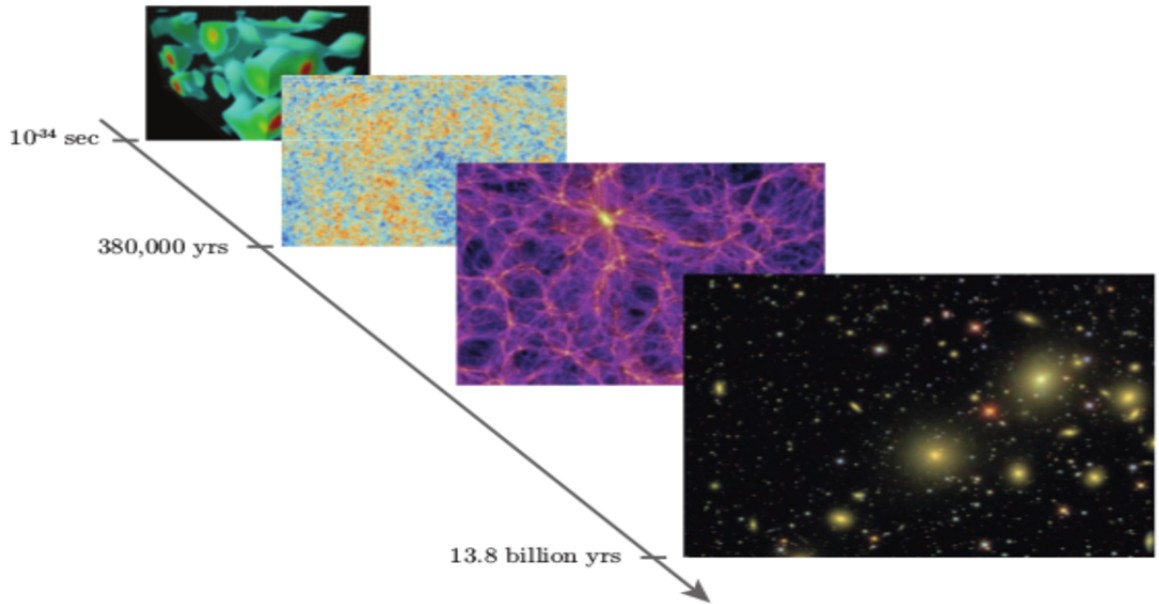


Figure 2.2: Illustration of a growing Universe, at the age of 380,000 years we observed CMB as seen by Planck and WMAP and the galaxy distribution observed today [17].

Topology of an expanding space

The concept of an expanding Universe was/is still a challenging concept in cosmology. Even though observations fully support idea, one question remains: where is the Universe expanding into?. Sometimes the words "Universe" and "space" are used interchangeably, although they have different meanings.

- **Space** is a mathematical concept that stands for the three-dimensional manifold into which our respective positions are embedded [18].
- **Universe** refers to everything that exists including the matter and energy in space, the extra-dimensions that may be wrapped up in various strings, and the time through which various events take place [18]. The expansion of space is in reference to this 3-D manifold only, that is, the description involves no structures such as extra dimensions or an exterior Universe [19].

In expanding space, we use co-moving coordinates because proper distances are dynamical quantities which change with time. In co-moving coordinates, the distances between all objects are fixed and the instantaneous dynamics of matter and light are determined by the normal physics of gravity and electromagnetic radiation, the time-evolution will be taken care by considering the Hubble law. Hubble's law tells us that galaxies that are further away are higher red-shifted than galaxies that are close. Some galaxies are blue-shifted relative to us (Milky Way galaxy) but the general overwhelming effect is that, galaxies recede faster from us the more distant they are [20], which supports the idea of an expanding Universe, see Fig. (2.3)

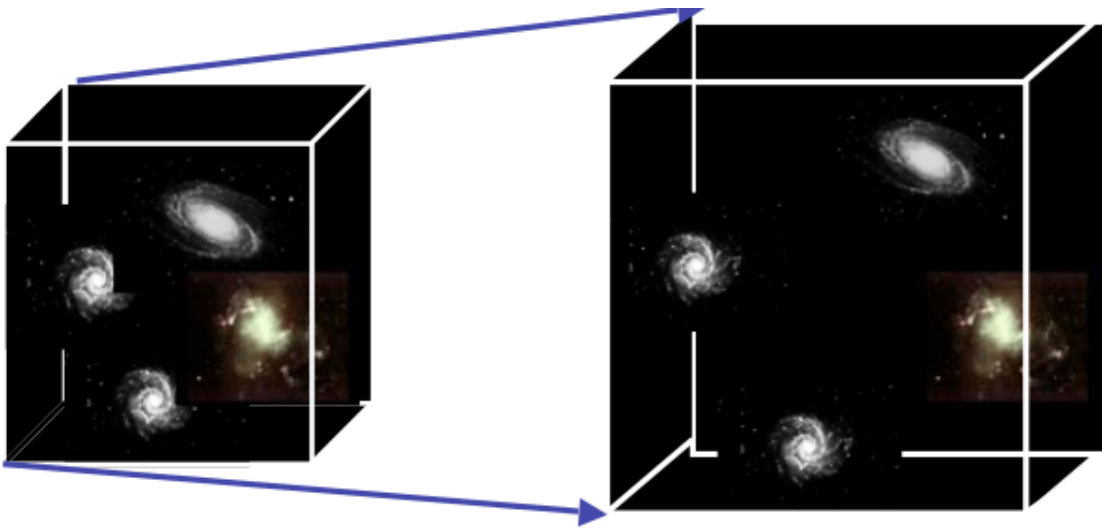


Figure 2.3: On large scales, galaxies are moving apart, with velocity proportional to distance. It is not galaxies moving through space but space is expanding, carrying the galaxies along. The galaxies themselves are not expanding.

2.2.2 Evidence from Supernova Type Ia

The first observational evidence which quantified the accelerated expansion of the cosmos was the measurement of the luminosity distance of Type Ia supernovae (SN Ia) [21]. Other experimental evidences range from CMB observations e.g. the Planck survey [22] to galaxy surveys e.g. the Dark Energy Survey (DES), Sloan Digital Sky Survey (SDSS) and the Baryon Oscillation Spectroscopic Survey (BOSS) [23]. Among the above experiments, the highest precision for observational data comes from the Planck satellite. The SN Ia experiment is still the most sensitive probe of dark energy. SN Ia studies were very sensitive to how the effects of dust are disentangled from intrinsic color variations.

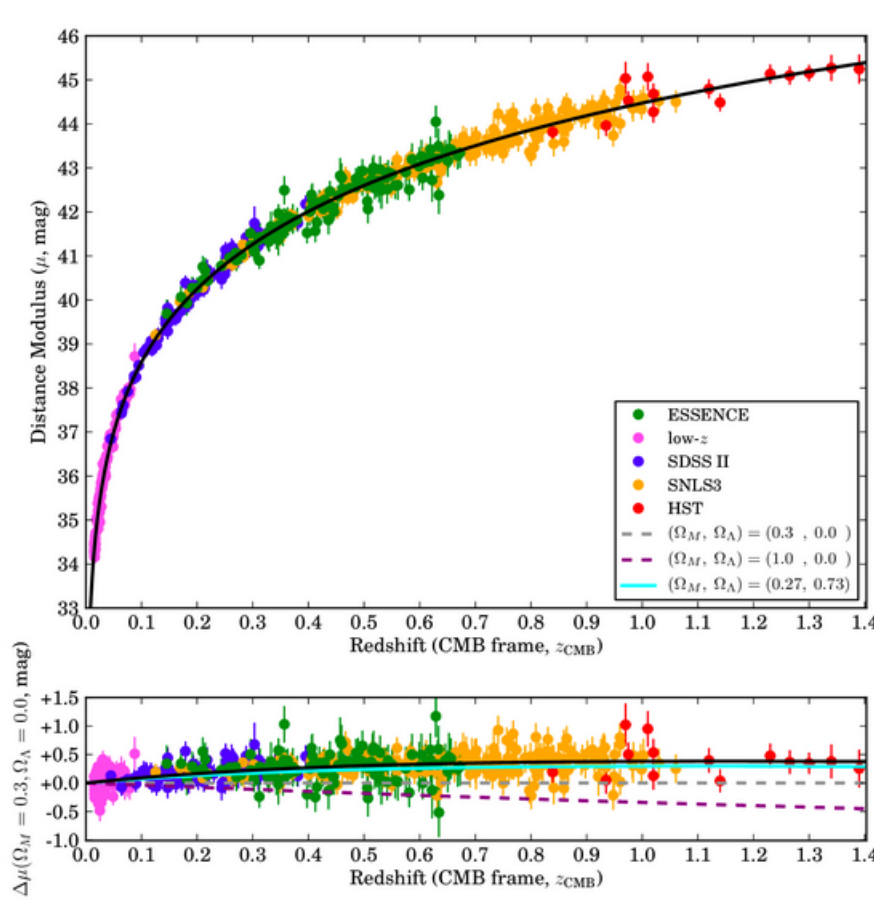


Figure 2.4: This shows a best-fit plot of Distance Modulus against the red-shift of galaxies fitted by data from different surveys [24].

2.2.3 Cosmic Microwave Background

Just after the Big Bang, the Universe expanded very quickly. This process was referred to as inflation. In the very early stages of the Universe, we had mainly hot photons and baryons coupled together in a plasma fluid. The main type of interaction that dominated was the Thomson scattering. This was called the radiation era. With time as the Universe cooled down, the photons could decouple themselves from the baryons and free stream through space time leaving the baryons behind and there was a leftover radiation emitted when the Universe stopped being a plasma (ionized gas state) and turned into a gas, this is known as Cosmic Microwave Background **CMB**. This happened about 380,000 years after the Big Bang and imprinted on it are traces of the seeds from which the stars and galaxies we can see today eventually formed.

The **CMB** measurements are a clear evidence supporting the idea of an expanding Universe [25], see Fig. (2.5).

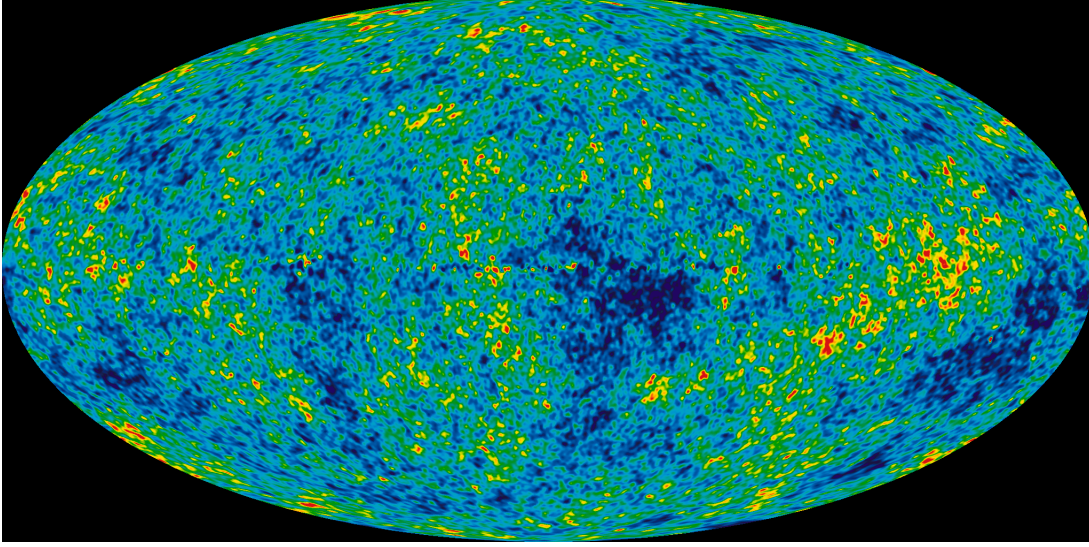


Figure 2.5: Illustration of homogeneity and isotropy of the observed Universe [26].

One of the implications of the cosmological principle is that, all parts of the space are causally connected in the past, meaning they may no longer be connected today [26]. Also since the laws of nature are assumed not to change over time, meaning what was observed in the past can be assumed to operate under the same physics even today's [25]. It is important to notice that the cosmological principle ignores proper motions arising from local condensations of matter [27].

2.2.4 Cosmological redshift

As the galaxies move apart, they emit light and the light emitted provides us with the information that we know about our Universe [28]. Because the Universe is expanding, the wavelength of the emitted light is lengthened as the expansion takes place and this is called cosmological redshift. The red-shifting of photons is inversely proportional to the photon momentum. But the physics tells us that the energy of mass-less particles decays with the expansion of the Universe, which implies that the momentum of a photon evolves as $a(t)^{-1}$ and the wavelength scales as $a(t)$ [17].¹ The light emitted at time t_1 with the wavelength λ_1 will be observed at t_0 with the wavelength

$$\lambda_0 = \frac{a_0}{a_1} \lambda_1, \quad (2.6)$$

where a_0 is the scale factor at t_0 , (i.e., today) and a_1 is the scale factor at t_1 . Since $\lambda_0 > \lambda_1$ it implies that $a_0 > a_1$.

The above results can be achieved either by the use of quantum mechanical (freely propagating

¹The scale factor parameter $a(t)$ is discussed more in Section(2.2.5).

photons) as done here or by using classical electromagnetic waves.

Now define the redshift parameter as the fractional shift in wavelength of a photon emitted by galaxy at time t_1 and observed on Earth today [17]:

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1}. \quad (2.7)$$

By using Eq. (2.6) in Eq. (2.7), we obtain

$$z + 1 = \frac{a_0}{a_1}. \quad (2.8)$$

For $a_0 = 1$, i.e., today Eq. (2.8) can be written as

$$z + 1 = \frac{1}{a_1}. \quad (2.9)$$

It is at this point when Edwin Hubble realized that for nearby sources, we can actually expand a_1 in power series as [17]

$$a_1 = a_0 [1 + (t_0 - t_1)H_0 + \dots], \quad (2.10)$$

where the parameter H_0 is defined as the Hubble constant given by

$$H_0 \equiv \frac{\dot{a}_0}{a_0}. \quad (2.11)$$

The Eq. (2.8) gives us

$$z = H_0(t_0 - t_1) + \dots \quad (2.12)$$

$(t_0 - t_1)$ has the unit of time. we therefore observed that the redshift is linearly increasing with distance, i.e.,

$$z \simeq dH_0. \quad (2.13)$$

H_0 normalises everything and the speed of light $c = 1$, then the measured value of Hubble constant is

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad (2.14)$$

where the measured value of h today according to Planck observations of 2018 is [29]

$$h = 0.674 \pm 0.05 \quad (2.15)$$

2.2.5 Evolution of a cosmic scale factor

The evolution of scale factor is a dynamical question, which arise by solving Einstein's field equations for the homogeneous and isotropic Universe, i.e., the Friedmann equations Eq. (2.1) [30].

Since the Universe is not only expanding, its expansion rate is increasing over time. This implies that the second derivative of a scale factor $a(t)$ is positive, meaning its first derivative is increasing over time as well, i.e., $\ddot{a}(t) > 0$.

This also means that for any given object moving away from us its receding velocity increases with time and its scale factor is defined as [30]

$$a(t) = \frac{1}{1+z} \quad (2.16)$$

By solving Eq. (2.17), we can show how the evolution of scale factor during different epochs of the Universe is:

Let us start by dark-energy dominated Universe; we consider

$$H^2 = \frac{8\pi G \rho_\Lambda}{3}, \quad (2.17)$$

where $\rho_\Lambda = \Lambda/8\pi G$.

Then we can use ρ_Λ in Eq. (2.17) and solve for H , results are

$$H = \sqrt{\frac{\Lambda}{3}}. \quad (2.18)$$

We can then substitute Eq. (2.11) in Eq. (2.18) and solve to obtain

$$a(t) \propto e^{\sqrt{\frac{\Lambda}{3}}t}, \quad (2.19)$$

which can be written in the form

$$a(t) \propto e^{Ht} \quad (2.20)$$

This shows an exponential growth in the scale factor, characterising a rapid expansion of the Universe.

For matter dominated Universe, we consider

$$H^2 = \frac{8\pi G \rho_{m0}}{3a^3}, \quad (2.21)$$

where

$$\rho = \rho_{m0}a^{-3} \quad \text{and} \quad H_0^2 = \frac{8\pi G}{3} \frac{\rho_{m0}}{a_0^3} \quad (2.22)$$

and the 0 represents today's values .

Using the fact that $a_0 = 1$ in this era, it is trivial to show

$$H^2 = H_0^2 a^{-3}, \quad (2.23)$$

which can be expressed as

$$a(t)[\dot{a}(t)]^2 = H_0^2. \quad (2.24)$$

Integrating both sides we get

$$a(t) \propto t^{2/3} \quad (2.25)$$

Similarly for the radiation dominated Universe, i.e., $\rho = \rho_r$, the same manipulation applies and the results obtained are

$$a(t) \propto t^{1/2}. \quad (2.26)$$

It turns out that the scale $a(t)$ is one of the fundamental cosmological parameters since it can be used to calculate the age of the observable Universe [14].

To show this, we start by

$$\dot{a}(t) = \frac{da(t)}{dt} \quad (2.27)$$

where in this case ($\dot{} \equiv d/dt$) and we can expand Eq. (2.27) and manipulate to get

$$dt = \frac{a(t)da(t)}{a(t)^2 H}. \quad (2.28)$$

We then consider our FLRW model equation

$$H = H_0 \sqrt{\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_k a^{-2} + \Omega_\Lambda}. \quad (2.29)$$

where the parameters in Eq. (2.29) are dimensionless densities, which are given by:

$$\Omega_m = \frac{8\pi G \rho_{m0}}{3H_0^2}, \quad \Omega_r = \frac{8\pi G \rho_{r0}}{3H_0^2}, \quad \Omega_\Lambda = \frac{8\pi G \rho_\Lambda}{3H_0^2}, \quad \Omega_k = -\frac{k}{H_0^2 a^2} \quad (2.30)$$

It follows from Eq. (2.28) and Eq. (2.29) that

$$dt = \frac{a(t)da(t)}{\sqrt{\Omega_m a + \Omega_r + \Omega_k a^2 + \Omega_\Lambda a^4}}, \quad (2.31)$$

this can be integrated to obtain

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{a(t)da(t)}{\sqrt{\Omega_m a + \Omega_r + \Omega_k a^2 + \Omega_\Lambda a^4}}, \quad (2.32)$$

and with the correct values of cosmological parameters, this gives the known value of $t_0 \approx 13.8$ billion years

2.2.6 Summary for Chapter 2

This chapter discussed mainly about our standard cosmology, addressing how it is constructed from the so-called cosmological principle and issues facing it. The observational tests were studied, among them the SN1a particularly, the CMB specifically how they confirm the Big Bang theory, the cosmological redshift was also studied together with the evolution of scale factor $a(t)$.

THE FRIEDMANN-LEMAÎTRE-ROBERTSON-WALKER UNIVERSE

3.1 The Friedmann-Lemaître-Robertson-Walker Metric

The FLRW Universe exist under the two main assumptions, **(a)** under the homogeneity and an isotropy assumption. and **(b)** that the spatial component of the metric can be time-dependent. The generic metric which meets these conditions is described by the following 4-D line element [31]:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi) \right), \quad (3.1)$$

where (r, θ, ϕ) are the co-moving radial and transverse coordinates respectively.

When consider a flat Universe and therefore we set $k = 0$, so the metric in Eq. (3.1) becomes

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & r^2 a^2 \sin^2 \theta \end{pmatrix}. \quad (3.2)$$

3.1.1 The Friedmann Equations

The Friedmann-Lemaître-Robertson-Walker metric is an exact solution of Einstein's field equations that describes our background spacetime which in this case results in a spatial flat Universe [10]. In 1929 Einstein developed the field equations for gravity in his GR and these equations explain the origin of gravity as the result of a distortion in space-time by the presence of matter and they are written as Eq. (1.2) [6].

The Ricci tensor $R_{\mu\nu}$ is a measure of curvature of space-time and it is given by Eq. (1.13), which depends on the Christoffel symbol [6].

R is the Ricci scalar and it is obtained by contracting the Ricci tensor as follows,

$$R = R_{\mu\nu}g^{\mu\nu}, \quad (3.3)$$

and $T_{\mu\nu}$ is the stress-energy momentum of a fluid and it is given by

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}. \quad (3.4)$$

where ρ is the total energy density, p is the pressure and $U_\mu = (-1, \mathbf{0})$ is the co-moving 4-velocity vector. For the FLRW metric given in Eq. (3.2) the non-zero components of the Christoffel symbol are,

$$\begin{aligned} \Gamma_{11}^0 &= a\dot{a}, & \Gamma_{22}^0 &= a\dot{a}r^2, \\ \Gamma_{33}^0 &= a\dot{a}r^2 \sin^2\theta, & \Gamma_{01}^1 &= \frac{\dot{a}}{a}, \\ \Gamma_{11}^1 &= 0, & \Gamma_{22}^1 &= -r, \\ \Gamma_{33}^1 &= -r \sin^2\theta, & \Gamma_{33}^2 &= -\sin\theta \cos\theta, \\ \Gamma_{12}^2 &= \Gamma_{13}^3 = \frac{1}{r}, & \Gamma_{23}^3 &= \cot\theta, \end{aligned} \quad (3.5)$$

Note that $\Gamma_{01}^1 = \Gamma_{02}^2 = \Gamma_{03}^3$ and therefore the non-zero components of the Ricci scalar are given by

$$\begin{aligned} R_{00} &= -3(\dot{H} + H^2), & R_{11} &= a^2(\dot{H} + 3H^2), \\ R_{22} &= a^2r^2(\dot{H} + 3H^2), & R_{33} &= a^2r^2 \sin^2\theta(\dot{H} + 3H^2). \end{aligned} \quad (3.6)$$

The Ricci scalar is calculated using Eq. (3.5) as

$$\begin{aligned} R &= R_{00}g^{00} + R_{11}g^{11} + R_{22}g^{22} + R_{33}g^{33}, \\ &= 6\dot{H} + 12H^2. \end{aligned} \quad (3.7)$$

The non-zero components of the stress-energy momentum tensor are

$$\begin{aligned} T_{00} &= \rho, & T_{11} &= p a^2, \\ T_{22} &= p a^2 r^2, & T_{33} &= p a^2 r^2 \sin^2\theta. \end{aligned} \quad (3.8)$$

The time-time component of Eq. (1.21). gives the first Friedmann equation and the space-space component gives the acceleration equation as

$$H^2 + \dot{H} = -\frac{4\pi G}{3}(1 + 3w)\rho \quad (3.9)$$

If the Universe is purely dominated by dark energy, $\rho = \rho_\Lambda$ and Eq.(2.21) becomes

$$H^2 = \frac{8}{3}\pi G \rho_\Lambda. \quad (3.10)$$

3.1.2 Conservation Equations

The conservation laws of energy (mass) and momentum can be obtained as:

$$\nabla_\nu T^{\mu\nu} = 0. \quad (3.11)$$

By expanding the LHS and taking only the time-time component, we can derive the conservation equation for energy and this equation is also known as the continuity equation and is given by [6]

$$\dot{\rho} + 3H(1+w)\rho = 0. \quad (3.12)$$

The general solution of Eq. (3.14) is

$$\rho \propto a^{-3(1+w)}. \quad (3.13)$$

For matter dominated phase, we have $w \approx 0$ and thus

$$\rho_m \propto a^{-3}. \quad (3.14)$$

For radiation dominated phase, we have $w \approx 1/3$ and thus

$$\rho_r \propto a^{-4}. \quad (3.15)$$

For Λ CDM model, ρ_Λ will be a constant whereas for interacting dark energy model, it is described by [32]

$$w_{de} = [w_0 a + w_e(1-a)]. \quad (3.16)$$

where w_{de} is the dark energy equation of state parameter, w_e is the early-time value of w_{de} and w_0 is the late-time value of w_{de} .

3.1.3 Cosmological parameters

We start with Friedmann Eq.(2.1) and aim to re-write it in dimensionless form, we note that $a(t)$ is arbitrary constant, which is taken to be $a_0 = 1$ at the present time. Then in dimensionless form Eq.(2.1) can be written as

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1, \quad (3.17)$$

where dimensionless densities parameters in Eq. (3.17) are given by Eq. (2.30).

It follows from Eq. (3.14), Eq. (3.15) and Eq. (2.20) that

$$\Omega_m = \Omega_{m0} a^{-3} \left(\frac{H_0}{H} \right)^2, \quad \Omega_r = \Omega_{r0} a^{-4} \left(\frac{H_0}{H} \right)^2, \quad \Omega_\Lambda = \Omega_{\Lambda0} \left(\frac{H_0}{H} \right)^2 \quad (3.18)$$

The parameter H_0 is defined as the Hubble rate today, similarly Ω_{m0} , Ω_{r0} , Ω_Λ are defined as density parameters for matter, radiation and cosmological constant respectively, evaluated today.

3.2 Cosmological perturbations

3.2.1 Linear perturbation theory

According to the cosmological principle, the Universe should be homogeneous and isotropic on large scales. However, that is not the case on small scales as the inhomogeneity exists. The isotropy of the CMB on smaller scales is an imprint of the homogeneity of the matter distribution at the time of recombination [33]. Hence it is fundamentally reasonable to work under the idea that the present structure of the Universe originates from the growth of initially small cosmological perturbations. The linear perturbation theory is the fundamental theory which explains the formation of structure in the Universe [17]. Studying the evolution of these fluctuations (perturbations) is essential, as it helps us to understand the origins of the structure formation. Let us consider the background Universe (unperturbed metric), which can be written as

$$\bar{ds}^2 = -dt^2 + a^2 \delta_{ij} dx^i dx^j. \quad (3.19)$$

The term dt^2 is the cosmic time which can be transformed and written in terms of conformal time as $d\eta = dt/a(t)$. We can now introduce a conformal time into the metric, then Eq. (3.19) becomes

$$\bar{ds}^2 = a^2(\eta) \left[-d\eta^2 + \delta'_{ij} dx^i dx^j \right]. \quad (3.20)$$

where the term $d\eta^2$ is the time-time term and $dx^i dx^j$ is a space-space term. By perturbing these two (time-time and space-space terms), we introduce two new degrees of freedom (scalars). Hence the metric takes the form

$$ds^2 = a^2(\eta) \left[-(1 + 2A)d\eta^2 + 2B_i dx^i + (\delta_{ij} + h_{ij}) dx^i dx^j \right], \quad (3.21)$$

where A, B_i and h_{ij} are functions of space and time.

Consider the situations where anisotropic stress is not taken into account, the perturbed metric takes the form

$$ds^2 = a^2(\eta) \left[-(1 + 2\Psi)d\eta^2 + (1 - 2\Phi)\delta_{ij} dx^i dx^j \right], \quad (3.22)$$

where Φ and Ψ are Bardeen variables and they are defined as [34],

$$\Psi \equiv A + \mathcal{H}(B - E') + (B - E')' \quad \text{and} \quad \Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E,$$

(A, B, C, E) are scalars, $\mathcal{H} \equiv aH$, is the Hubble parameter given in terms of conformal time and the prime denote derivative with respect to conformal time. The Bardeen's variable are used to avoid the gauge problem since they do not transform under a change of coordinates.

For $B = E = 0$, we have

$$g_{\mu\nu} = a^2 \begin{pmatrix} -(1+2\Psi) & 0 \\ 0 & +(1-2\Phi)\delta_{ij} \end{pmatrix}. \quad (3.23)$$

By the use of Eq. (1.13) and the Einstein field equations Eq. (1.21), the linear perturbation equations are obtained as follows [17]:

$$\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \rho \delta = \frac{3}{2}\mathcal{H}^2\delta \quad (3.24)$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2 (\bar{\rho} + \bar{P})v = -\frac{3}{2}\mathcal{H}^2(1+w)v \quad (3.25)$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta \quad (3.26)$$

Then we have

$$\nabla^2(\Phi - \Psi) = \Pi_{ij}. \quad (3.27)$$

Therefore the spatial part of the anisotropic stress tensor from Eq. (3.27) can be chosen to be traceless, $\Pi_{ij} = 0$. Without loss of generality, we can then set $\Pi_{00} = \Pi_{i0} = 0$ which implies $\Phi = \Psi$. The Bardeen variables are gauge-invariant [17], thus these potentials are equal to two nonzero metric perturbations in the conformal-Newtonian gauge.

3.2.1.1 Perturbation in the Curvature Tensors

From Eq. (3.23), we can get the connection coefficients that relate the Ricci tensor with the Einstein tensor:

$$\begin{aligned} \Gamma_{00}^0 &= \frac{a'}{a} + \Phi', & \Gamma_{0k}^0 &= \Phi_{,k}, & \Gamma_{ij}^0 &= \frac{a'}{a}\delta_{ij} - \left[2\frac{a'}{a}(\Psi + \Phi) + \Psi'\right]\delta_{ij}, \\ \Gamma_{00}^i &= \Phi_{,i}, & \Gamma_{0j}^i &= \frac{a'}{a}\delta_j^i - \Psi'\delta_j^i, & \Gamma_{kl}^i &= -(\Psi_{,l}\delta_j^i + \Psi_{,k}\delta_j^i) + \Psi_{,i}\delta_{kl}, \end{aligned} \quad (3.28)$$

and the sums

$$\Gamma_{0\alpha}^\alpha = 4\frac{a'}{a} + \Phi' - 3\Psi, \quad \Gamma_{i\alpha}^\alpha = \Phi_{,i} - 3\Psi'_{,i}, \quad (3.29)$$

where the higher order terms are dropped and only zeroth and first orders terms are considered and separated into the background and perturbation:

$$\Gamma_{\beta\gamma}^\alpha = \bar{\Gamma}_{\beta\gamma}^\alpha + \delta\Gamma_{\beta\gamma}^\alpha \quad (3.30)$$

where

$$\begin{aligned} \bar{\Gamma}_{00}^0 &= \mathcal{H}, & \bar{\Gamma}_{0k}^0 &= 0, & \bar{\Gamma}_{ij}^0 &= \mathcal{H}\delta_{ij}, \\ \bar{\Gamma}_{00}^i &= 0, & \bar{\Gamma}_{0j}^i &= \mathcal{H}\delta_j^i, & \bar{\Gamma}_{kl}^i &= 0, \end{aligned} \quad (3.31)$$

and the small perturbations are given as

$$\begin{aligned}\delta\Gamma_{00}^0 &= \Phi', & \delta\Gamma_{0k}^0 &= \Phi_{,k}, & \delta\Gamma_{ij}^0 &= -\left[2\mathcal{H}(\Psi + \Phi) + \Psi'\right]\delta_{ij}, \\ \delta\Gamma_{00}^i &= \Phi_{,i}, & \delta\Gamma_{0j}^i &= -\Psi'\delta_j^i, & \delta\Gamma_{kl}^i &= -\left(\Psi_{,l}\delta_j^i + \Psi_{,k}\delta_j^i\right) + \Psi_{,i}\delta_{kl},\end{aligned}\tag{3.32}$$

The Ricci tensor is

$$\begin{aligned}R_{\mu\nu} &= \Gamma_{\nu\mu,\alpha}^\alpha - \Gamma_{\alpha\mu,\nu}^\alpha + \Gamma_{\alpha\beta}^\alpha \Gamma_{\nu\mu}^\beta - \Gamma_{\nu\beta}^\alpha \Gamma_{\alpha\mu}^\beta, \\ &= \bar{R}_{\mu\nu} + \delta\Gamma_{\nu\mu,\alpha}^\alpha - \delta\Gamma_{\alpha\mu,\nu}^\alpha + \bar{\Gamma}_{\alpha\beta}^\alpha \delta\Gamma_{\nu\mu}^\beta + \bar{\Gamma}_{\nu\mu}^\beta \delta\Gamma_{\alpha\beta}^\alpha - \bar{\Gamma}_{\nu\beta}^\alpha \delta\Gamma_{\alpha\mu}^\beta - \bar{\Gamma}_{\alpha\mu}^\beta \delta\Gamma_{\nu\beta}^\alpha.\end{aligned}\tag{3.33}$$

Finally, we obtain the following:

$$\begin{aligned}R_{00} &= -3\mathcal{H}' + 3\Psi'' + \nabla^2\Phi + 3\mathcal{H}(\Phi' + \Psi'), \\ R_{0i} &= 2\left(\Psi' + \mathcal{H}\Phi\right)_{,i}, \\ R_{ij} &= \left(\mathcal{H}' + 2\mathcal{H}^2\right)\delta_{ij} + \left[-\Psi'' + \nabla^2\Psi - \mathcal{H}(\Phi' + 5\Psi') - \left(2\mathcal{H}' + 4\mathcal{H}^2\right)(\Phi + \Psi)\right]\delta_{ij} + (\Psi - \Phi)_{,ij}.\end{aligned}\tag{3.34}$$

The goal is to derive the Ricci scalar and the Einstein tensor, here we need to raise index to get R_ν^μ . Please note that we cannot raise the index of the background and perturbation parts separately [4], since

$$\begin{aligned}R_\nu^\mu &= g^{\mu\alpha}R_{\alpha\nu}, \\ &= \left(\bar{g}^{\mu\alpha} + \delta g^{\mu\alpha}\right)\left(\bar{R}_{\alpha\nu} + \delta R_{\alpha\nu}\right), \\ &= \bar{R}_\nu^\mu + \delta g^{\mu\alpha}\bar{R}_{\alpha\nu} + \bar{g}^{\mu\alpha}\delta R_{\alpha\nu}.\end{aligned}\tag{3.35}$$

When solving Eq. (3.35), the following results are obtained:

$$\begin{aligned}R_0^0 &= \frac{3\mathcal{H}'}{a^2} + \frac{1}{a^2}\left[-3\Psi'' - \nabla^2\Phi - 3\mathcal{H}(\Phi' + \Psi') - 6\mathcal{H}'\Phi\right], \\ R_i^0 &= -\frac{2}{a^2}\left(\Psi' + \mathcal{H}\Phi\right)_{,i}, \\ R_0^i &= -R_i^0, \\ R_j^i &= \frac{1}{a^2}\left(\mathcal{H}' + 2\mathcal{H}^2\right)\delta_j^i + \frac{1}{a^2}\left[-\Psi'' + \nabla^2\Phi - \mathcal{H}(\Phi' + 5\Psi') - \left(2\mathcal{H}' + 4\mathcal{H}^2\right)\Phi\right]\delta_{ij}.\end{aligned}\tag{3.36}$$

Summing up these for the curvature scalar

$$\begin{aligned}R &= R_0^0 + R_i^i \\ &= \frac{6}{a^2}\left(\mathcal{H}' + \mathcal{H}^2\right) + \frac{1}{a^2}\left[-6\Psi'' + 2\nabla^2(2\Psi - \Phi) - 6\mathcal{H}(\Phi' + 3\Psi') - 12(\mathcal{H}' + \mathcal{H}^2)\Phi\right].\end{aligned}\tag{3.37}$$

At this point we have all the material needed to construct the Einstein tensor:

$$\begin{aligned}G_0^0 &= R_0^0 - \frac{1}{2}R = \frac{3}{a^2}\mathcal{H}^2 + \frac{1}{a^2}\left[-2\nabla^2\Psi + 6\mathcal{H}\Psi' + 6\mathcal{H}^2\Phi\right], \\ G_i^0 &= R_i^0, \\ G_0^i &= R_0^i = -G_i^0, \\ G_j^i &= R_j^i - \frac{1}{2}\delta_j^i R, \\ &= \frac{1}{a^2}\left[2\Psi'' + \nabla^2(\Psi - \Phi) + \mathcal{H}(2\Phi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2)\Phi\right]\delta_j^i + \frac{1}{a^2}(\Psi - \Phi)_{,ij}.\end{aligned}\tag{3.38}$$

The background is always written first in all of these quantities followed by the perturbation part. \bar{R}_ν^μ and \bar{G}_ν^μ are diagonal, the off diagonals quantities contain perturbations and we have

$$\delta R_i^0 = \delta G_i^0 \quad (3.39)$$

Finally in the following section, I will show how perturbation occur in the energy-momentum tensor of a perfect fluid.

3.2.1.2 Perturbation in the Energy-Momentum Tensor

Consider the background energy-momentum tensor

$$\begin{aligned} \bar{T}^{\mu\nu} &= (\bar{\rho} + \bar{p}) \bar{u}^\mu \bar{u}^\nu + \bar{p} \bar{g}^{\mu\nu}, \\ \bar{T}_\nu^\mu &= (\bar{\rho} + \bar{p}) \bar{u}^\mu \bar{u}_\nu + \bar{p} \delta_\nu^\mu. \end{aligned} \quad (3.40)$$

When we consider the homogeneity ($\bar{\rho} = \bar{\rho}(\eta)$ and $\bar{p} = \bar{p}(\eta)$) and similarly for isotropy, the fluid is at rest, i.e., $\bar{u}^i = 0$ which implies that $\bar{u}^\mu = (\bar{u}^0, 0, 0, 0)$ in the background Universe [4].

But we have

$$\begin{aligned} \bar{u}_\mu \bar{u}^\mu &= \bar{g}_{\mu\nu} \bar{u}^\mu \bar{u}^\nu, \\ &= a^2 \eta_{\mu\nu} \bar{u}^\mu \bar{u}^\nu, \\ &= -a^2 (\bar{u}^0)^2, \\ &= -1. \end{aligned} \quad (3.41)$$

which results in the following

$$\bar{u}^\mu = \frac{1}{a} (1, \vec{0}), \quad \bar{u}_\mu = a (-1, \vec{0}) \quad (3.42)$$

For the perturbed Universe the energy tensor is given as

$$T_\nu^\mu = \bar{T}_\nu^\mu + \delta T_\nu^\mu. \quad (3.43)$$

Similarly the energy tensor is perturbed like a metric, it also has 10 degrees of freedom. four are coming from gauge contribution and the other 6 are physical, which can be divided into three components the tensor, scalar and vector [4]. Likewise the perturbation can be also divided into two parts, the perfect fluid and non-perfect fluid, with 5+5 degrees of freedom. The δT_ν^μ contains the perfect fluid degrees of freedom which keep T_ν^μ in the perfect fluid form

$$T_\nu^\mu = (\rho + p) u^\mu u_\nu + p \delta_\nu^\mu. \quad (3.44)$$

Thus the density perturbation, pressure perturbation and velocity perturbation are

$$\rho = \bar{\rho} + \delta\rho, \quad p = \bar{p} + \delta p, \quad u^i = \bar{u}^i + \delta u^i = \delta u^i \equiv \frac{v_i}{a} \quad (3.45)$$

Note that δu^0 is not an independent degree of freedom, because of the constraint $u_\mu u^\mu = -1$.

where v_i represents the velocity perturbation, which is equal to the coordinate velocity for first order perturbation, i.e., $v_i \equiv a u^i$.

3.2.1.3 Cosmological perturbations in general

Cosmological perturbations are contained and described by the line element Eq. (3.19), which can be decoupled into background and perturbed parts as

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu + \delta g_{\mu\nu} dx^\mu dx^\nu, \quad (3.46)$$

where the bar represents background and $\mu\nu$, they represents 4-dimensional space-time, refer to Section (0.1) and $\bar{g}_{\mu\nu}$ is the homogeneous FLRW background metric. Lastly the $\delta g_{\mu\nu}$ describes the "small perturbations" metric. The perturbation $\delta g_{\mu\nu}$ can be divided into 3 parts of perturbations, the vector, tensor and scalar, i.e.,

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}^V + \delta g_{\mu\nu}^T + \delta g_{\mu\nu}^S. \quad (3.47)$$

Tensor perturbations $\delta g_{\mu\nu}^T$ produce gravitational waves and these gravitational waves do not couple to energy density and pressure inhomogeneity, they propagate freely. The Vector perturbation part $\delta g_{\mu\nu}^V$ explains the cosmological expansion of the Universe, they can be neglected today. Finally the Scalar part of perturbation $\delta g_{\mu\nu}^S$ is known to be responsible for the growing of inhomogeneity which gives rise to the large scale structure and the CMB anisotropies that can be observed today. We now show how each type of perturbation is implemented.

- **Vector perturbations**

These perturbations can be represented by two divergenceless three-vectors F_i and S_i as follows:

$$\delta g_{\mu\nu}^V = -a^2(\eta) \begin{pmatrix} 0 & -S_i \\ -S_i & (F_{i,j} + F_{j,i}) \end{pmatrix}. \quad (3.48)$$

By using Einstein's convention, we can easily show that the divergenceless conditions are $F_{,i}^i = S_{,i}^i = 0$, the same applies when the upper indices are lowered and the down ones raised by the use of spatial part of a spatially flat background metric tensor and its inverse, i.e., δ_{ji} and δ^{ji} . In these perturbations, the gauge-invariant quantity is given by $\mathbf{S}_i = S_i + F'_i$. Vector perturbations are divergence-free. For instance one can distinguish an intrinsically vector part of the metric perturbation, which is S_i . The prime that is in the divergenceless vector perturbation derivative F'_i denotes the derivative with respect to conformal time.

- **Scalar perturbations**

The most general form for $\delta g_{\mu\nu}^S$ perturbation is given by four scalar functions, (Φ, Ψ, E and B) of space-time coordinates as follows:

$$\delta g_{\mu\nu}^S = a^2(\eta) \begin{pmatrix} 2\Phi & -B_{,i} \\ -B_{,i} & 2(\Psi\delta_{ij} - E_{,ij}) \end{pmatrix}. \quad (3.49)$$

Here $i, j = 1, 2, 3$.

• **Tensor perturbations**

These perturbations are given by a symmetric three-tensor h_{ij} and satisfying the following conditions:

$$h_i^i = h_{,j}^{ij} = 0. \quad (3.50)$$

Note: This means h_{ij} does not change under gauge transformation, unlike vectors and scalars, hence the metric $\delta g_{\mu\nu}^T$ is given by

$$\delta g_{\mu\nu}^T = -a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}. \quad (3.51)$$

We observed that the number of independent functions that are introduced by using $\delta g_{\mu\nu}$ is ten: two three-vectors for vector perturbations with one constraint, four scalar functions responsible for scalar perturbations and lastly we have one symmetric three-tensor with four conditions for tensor perturbations. If we recall the number of independent components of $\delta g_{\mu\nu}$ as a 4x4 symmetric tensor, we see the coincidence with the number of independent functions introduced by $\delta g_{\mu\nu}$ without losing any generality.

3.2.2 Gauge Problem

The theory of cosmological perturbations is well known to be a complicated study, this is mainly caused by the issue of Gauge invariance. Metric perturbations are gauge-dependent [1].

Points in space-time manifold are designated by the coordinates (t, x) , they do not have any physical meaning other than just being labels. Performing a small-amplitude transformation of these space-time coordinates (i.e., gauge transformation), we can easily introduce "fabricated" fluctuations in an homogeneous and isotropic Universe.

As mentioned previously, perturbations are gauge-dependent meaning gauge-transformation could give rise to two apparently different perturbations which can lead to different relics of the Universe while representing the same physical perturbation [1]. This is when the gauge problem starts, causing a debate/confusion which led to an idea by Bardeen. Bardeen introduced gauge-invariant quantities that are explicitly invariant under infinitesimal coordinates transformations.

Consider an infinitesimal coordinates transformation:

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu(x), \quad (3.52)$$

where \tilde{x}^μ represents the set of new coordinates and $\xi^\mu(x)$ is the coordinate transformation parameter.

In the new coordinates \tilde{x}^μ , the metric Eq.(1.3) can be presented as

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \mathcal{L}_\xi g_{\mu\nu}(x) + \mathcal{O}(\xi^2), \quad (3.53)$$

where $\mathcal{L}_\xi g_{\mu\nu}(x)$ is the Lie derivative of a twice covariant tensor $g_{\mu\nu}$ with respect to ξ and $\mathcal{O}(\xi^2)$ is the higher-order derivatives term.

This proves that two metrics $\tilde{g}_{\mu\nu}(x)$ and $g_{\mu\nu}(x)$ differing in Lie derivative represent the same physical perturbation. Let us use coordinate transformation to show gauge-invariance, we start with parameters (ξ^0, ξ^i) such that,

$$\begin{aligned}\tilde{\eta} &= \eta + \xi^0, \\ \tilde{x}^i &= x^i + \xi^i,\end{aligned}\tag{3.54}$$

where

$$\xi^i = \bar{\xi}^i + \xi_{,j} \delta^{ij}.\tag{3.55}$$

According to Helmholtz's theorem the parameter ξ^i can be decomposed into two physical part of coordinates, i.e., the $\bar{\xi}^i$ is given by a solenoidal part and $\xi_{,j} \delta^{ij}$ is an irrotational part. Therefore we can express the derivatives $d\eta, dx^i$ as well as $a(\eta)$ in terms of new coordinates \tilde{x}^μ as follows:

$$\begin{aligned}d\eta &= d\tilde{\eta} - \xi^{0'} d\tilde{\eta} - \xi_{,i}^0 dx^i, \\ dx^i &= d\tilde{x}^i - \xi^{i'} d\tilde{\eta} - \xi_{,j}^i d\tilde{x}^j, \\ &= d\tilde{x}^i - (\bar{\xi}^{i'} + \xi_{,j}^{i'} \delta^{ij}) d\tilde{\eta} - (\bar{\xi}_{,j}^i + \xi_{,kj} \delta^{ik}) d\tilde{x}^j \\ a(\eta) &= a(\tilde{\eta}) - \xi^0 a'(\tilde{\eta}).\end{aligned}\tag{3.56}$$

Now we can use the identities Eq.(3.56) in the Eq.(3.46) to obtain the metric with the aspect of the original line element, provided that Eq.(3.48), Eq.(3.49) and Eq.(3.51) transform as follows [35]:

- **Vector perturbations**

$$\begin{aligned}\tilde{F}_i &= F_i - \tilde{\xi}_i \\ \tilde{S}_i &= S_i - \tilde{\xi}_i\end{aligned}\tag{3.57}$$

We can see that only vector contribution $\tilde{\xi}_i$ is present.

- **Scalar perturbations**

$$\begin{aligned}\tilde{\Psi} &= \psi + \mathcal{H}\xi^0, \\ \tilde{\Phi} &= \phi - \mathcal{H}\xi^0 - \xi^{0'} \\ \tilde{B} &= B - \xi^0 - \xi' \\ \tilde{E} &= E - \xi\end{aligned}\tag{3.58}$$

Similarly we notice that only scalar contributions, ξ^0 and ξ are present.

- **Tensor perturbations**

$$\tilde{h}_{ij} = h_{ij} \quad (3.59)$$

which indeed is gauge-invariant. From the above results, one can extract two gauge-invariant quantities, for instance these could be [36]:

$$\begin{aligned} \Phi &= \phi + \frac{1}{a} \left[(B - E')a \right]', \\ \Psi &= \psi + \mathcal{H}(B - E'). \end{aligned} \quad (3.60)$$

By construction, it can be shown that $\phi = \tilde{\phi}$ and $\psi = \tilde{\psi}$, which are well known as Bardeen's potentials [17, 37]. There exist several possibilities for the gauge choice, among those, we have conformal-Newtonian gauge which is the one studied in this work. Also there is Synchronous gauge which arises under the condition $\phi = B = 0$ while the Longitudinal (conformal-Newtonian) gauge arises from the condition $B = E = 0$ [36]. Unlike Synchronous gauge here the coordinates are totally fixed since $E = 0$ which defines ξ uniquely.

3.2.3 Summary for Chapter 3

Chapter 3 was mainly about the Friedmann Universe and the theory of perturbations in cosmology, as well as the Gauge problem, i.e., the gauge invariant. This needs to be taken into consideration since the perturbations are gauge-dependent. The problem of inhomogeneity on small scales was addressed during the study, we also showed that in linear perturbation theory under the conformal-Newtonian gauge, the Bardeen potentials are equal to two nonzero metric perturbations.

f(R) THEORIES OF GRAVITY**4.1 *f(R)* Theories of gravity****4.1.1 Why modifying General Relativity**

The current standard model that explains the late accelerated expansion of the Universe is the Λ CDM model. Dark energy is speculated to be the cause of this expansion and is described by the Λ parameter. It dominates the energy budget of the Universe and yet, it is still a poorly understood species because it is not observed yet. A modest theoretical approach to understand its nature is via perturbation theory. General Relativity is by far still, the best theory of gravity that ever exists, it is in agreement with all the local gravity tests, nevertheless it does not explain the late time acceleration expansion of the Universe as well as the nature of inflationary physics. Astronomical observations show that there is not enough ordinary matter to account for the behavior of the galaxies and other massive astrophysical objects in the Universe. The above mentioned, became the reason to support the modification of GR.

One of the most basic and popular motivation behind this attempt is the issue or question of the well known cosmological constant problem. Supernovae studies have shown that Our Observable Universe is undergoing the acceleration phase on its expansion and the simplest way to explain the problem of accelerating Universe in GR was done by introducing the Lambda term (Λ) in the Einstein-Hilbert Action to account for the unknown energy "dark energy" since it is crucial to explain this late-time accelerating phase of our Universe. The measured value of the energy density of dark energy ($\rho_\Lambda \approx 10^{-30} \text{g/cm}^3$), is in disagreement with that of the Vacuum Energy, which is much bigger when compared [38].

The smallness of this measured value of (ρ_Λ) is puzzling, hence the attempt in modifying gravity from GR has grown this far. The main goal to achieve is to get the value of (ρ_Λ) as compared to the observed value, by modifying GR at the cosmological scales and to at least get something perceptible about the nature of dark energy and dark matter as well. Dark matter particles are invisible to light but endowed with gravity, however, none of our detectors or experiments have ever seen a dark matter particle directly, which leads to some doubt about the existence of dark matter [38].

4.1.2 Background Equations

Modified theories of gravity, $f(R)$ in particular, arise when one generalize the Lagrangian of the Hilbert-Einstein Action, from Eq. (1.9) to:

$$S_{f(R)} = \int d^4x \sqrt{-g} \left[\frac{R + f(R)}{16\pi G} \right], \quad (4.1)$$

where $f(R)$ is some generic function of the Ricci scalar [39].

One can derive the field equations by using the same idea that was used to obtain the Einstein field equations. In the context of $f(R)$, this is called metric $f(R)$ gravity.

Let us look at the basic steps of this process, since the main steps are the same as in Section (1.2.1) but there are some noticeable differences. By varying the modified Action Eq. (4.1) with respect to the metric and without treating the connection independently, one arrives at the modified field equations.

We start by variation of the determinant Eq.(1.19). Therefore, the variation of the Ricci scalar with respect to the inverse metric $g^{\mu\nu}$ is given by Eq.(1.17). Please refer to Section (1.2.1) for some missing steps.

From Eq. (1.17), the $\delta\Gamma_{\mu\nu}^\lambda$ is the difference of two connections, then it should transform as a tensor ,i.e.,

$$\delta\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\alpha} \left(\nabla_\mu \delta g_{\alpha\nu} + \nabla_\nu \delta g_{\alpha\mu} - \nabla_\alpha \delta g_{\mu\nu} \right). \quad (4.2)$$

We use Eq. (1.19) and substitute in Eq. (1.17) to obtain

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + g_{\mu\nu} \square \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu}, \quad (4.3)$$

where ∇_μ is the covariant derivative and $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the D'Alembert operator.

Finally the variation in the action

$$\begin{aligned} \delta S_{(f(R)+R)} &= \int \frac{1}{2\kappa} \left[\sqrt{-g} (\delta f(R) \delta R + \delta R) + (f(R) + R) \delta \sqrt{-g} \right] d^4x, \\ &= \int \frac{1}{2\kappa} \left[(f_R \delta R + \delta R) \sqrt{-g} - \frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} (f(R) + R) \right] d^4x, \\ &= \int \frac{1}{2\kappa} \sqrt{-g} \left[(f_R + 1) (R_{\mu\nu} \delta g^{\mu\nu} + g_{\mu\nu} \square \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu}) - \frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} (f(R) + R) \right] d^4x. \end{aligned} \quad (4.4)$$

where f_R is the first derivative of the generic function of the Ricci scalar with respect to Ricci scalar.

At this stage, we need to perform integration by parts on the 2^{nd} and 3^{rd} terms of the above equation to obtain the following:

$$\delta S_{f(R)} = \int \frac{1}{2\kappa} \sqrt{-g} \delta g^{\mu\nu} \left[(f_R + 1) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (f(R) + R) + (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) (f_R + 1) \right] d^4x. \quad (4.5)$$

But we know that the action is invariant under variations of the metric, we have $\delta S / \delta g^{\mu\nu} = 0$, which simplifies Eq. (4.5) to obtain

$$(f_R + 1) R_{\mu\nu} - \frac{1}{2} (f(R) + R) g_{\mu\nu} + [g_{\mu\nu} \square - \nabla_\mu \nabla_\nu] (f_R + 1) = \kappa T_{\mu\nu}, \quad (4.6)$$

and the full Modified Einstein's field Equations are

$$G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (f(R) + R) + R_{\mu\nu} (f_R + 1) - g_{\mu\nu} \square (f_R + 1) + (f_R + 1)_{;\mu\nu} = \kappa T_{\mu\nu}. \quad (4.7)$$

The energy-momentum tensor is given by Eq. (1.12) [40].

When we consider the flat space-time background, defined by FLRW metric, the equations read as follow [5]:

$$3\mathcal{H}'(1 + f_R) - \frac{1}{2}(R_0 + f_0)a^2 - 3\mathcal{H}f_R' = \kappa\rho_0 a^2, \quad (4.8)$$

and

$$(\mathcal{H}' + 2\mathcal{H}^2)(1 + f_R) - \frac{1}{2}(R_0 + f_0)a^2 - (\mathcal{H}f_R' + f_R'') = \kappa c_s^2 \rho_0. \quad (4.9)$$

Note that R_0 is a scalar curvature corresponding to the unperturbed metric, c_s^2 is a speed of sound, $f_0 \equiv f(R_0)$ and lastly $f_R \equiv df(R_0)/dR_0$, prime denotes the derivative with respect to the conformal time η and $\mathcal{H} = aH$ where H is a Hubble expansion rate, ρ_0 represents the background cosmological density and lastly a is a cosmic scale factor.

The two equations Eq. (4.8) and Eq. (4.9), are common, then we can simply combine them to get [5]

$$2(1 + f_R)(\mathcal{H}^2 - \mathcal{H}') + 2\mathcal{H}f_R' - f_R'' = \kappa\rho_0(1 + c_s^2)a^2. \quad (4.10)$$

and the conservation equation is

$$\rho_0' + 3(1 + c_s^2)\mathcal{H}\rho_0 = 0 \quad (4.11)$$

The first-order perturbation equation for $f(R)$ theories is obtained when using the perturbed metric Eq. (4.21) in conjunction with the perturbed energy-momentum tensor Eq. (4.22), provided that the background equations hold [5].

4.1.3 Viable models of $f(R)$

Some models of modified gravity theories have been ruled out, their inability to pass the Solar System tests was the main reason. This, of course, is based on the fact that General Relativity works very well in doing so. Some of these theories ,i.e., Tensor-vector-scalar-gravity (TeVeS) and Modified Newtonian Dynamics (MOND) are able to explain the galactic rotation curves [41], which is a big problem that General Relativity does not purely explain it without introducing the concept of dark matter [42]. There are some interesting questions pointing at modification of gravity, some of those are the following.

- Are modified theories of gravity credible?

Even though the investigations are currently running, the likely answer is Yes. Modified theories of gravity are credible, they are deep and equally fulfilling in explaining the evidence of dark sector of the Universe as dark matter particle approach [43]. As mentioned, the only empirical evidence of dark matter particle is provided, both in GR and Standard Model of particle physics, by introducing some kind of new physics and its fundamentals still can not be purely explain. Any viable dark matter theory has to be able to explain why the distribution of luminous matter in a galaxy predicts observed dark matter phenomena so tightly and with so little scatter in multiple respects such as rotation curves [44].

- Which modified gravity theories are viable?

Among all the series of modified theories of gravity, $f(R)$ theories are still the most reasonable extended theories of gravity to be considered. Because they are physically and mathematically reasonable, constructed from General Relativity with the aim to extend where GR could not cover, by just using a generic function of R , i.e., $f(R)$ instead of R in the Hilbert-Einstein's Action Eq. (4.1) [5]. It is logically and motivated to do so, since the Ricci scalar R is the only gravitational term that can be modified in the Hilbert-Einstein's Action. This is done by carefully taking care of the consequences that may arise, an example some class of $f(R)$ theories are defined by the $f(R)$ function instead of R as,

$$f(R) = R + g(R) \quad (4.12)$$

where $g(R)$ is the arbitrary function of R , which represents the "corrections" for GR. We can easily realize that when $g(R) = 0$, then GR is not affected. Therefore it makes sense that viable models of $f(R)$ should mimic General Relativity and are expected to do better than it at some level of scales. The following are two popular models of $f(R)$ class of modified theories of gravity that are considered to be viable, based on their ability to pass the Solar System tests well [45].

4.1.3.1 Starobinsky model

Quantum corrections are important for the early Universe, these lead to curvature-square corrections to Hilbert-Einstein's Action in form of $f(R)$ [46]. As required at high curvature values, Einstein's solutions in the presence of curvature square terms lead to an effective cosmological constant [47]. It was Starobinsky who proposed that at an early time the Universe went through an inflationary de Sitter era. The Starobinsky model is described by the following expression [48]:

$$f(R) = -c_1 m^2 \left[1 - \left(1 + \frac{R^2}{m^4} \right)^{-n} \right] \quad (4.13)$$

where $m^2 \approx \Omega_m H_0^2$, m and n are positive constants and c_1 is a free dimensionless model parameter. The Eq. (4.13) has been used several times in this field of cosmology such as [39, 45] and this model resolved the cosmology problems and led to specific predictions for the corrections to the microwave background radiation. Starobinsky inflation gives a prediction for the observables of the spectral tilt $n_s > -\frac{2}{N} + 1$ and the tensor-scalar ratio $r = \frac{12}{N^2}$, where N is the number of e-foldings since the horizon crossing.

As $50 < N < 60$, these are compatible with experimental data, with 2018 CMB data from the Planck satellite giving a constraint of $r < 0.064$ (95 % confidence) and $n_s = 0.9649 \pm 0.0042$ (68 % confidence).

4.1.3.2 The Hu-Sawicki model

Many $f(R)$ functions do face problems; the Hu-Sawicki model was designed to overcome those cosmological problems [49]. It is carefully constructed such that at high redshifts values, the general relativity theory is recovered and at a low redshifts values, it mimics the Λ CDM model very well, hence it is considered as a viable model [50]. The Hu-Sawicki model is described as follows [49]:

$$f(R) = -m^2 \left[\frac{c_1 \left(\frac{R}{m^2} \right)^n}{c_2 \left(\frac{R}{m^2} \right)^n + 1} \right], \quad (4.14)$$

where $m^2 \approx \Omega_m H_0^2$, m and n are positive constants, c_1 and c_2 are dimensionless model parameters.

In [50], it has been shown that c_1, c_2 are related to the energy densities for cosmological constant, Ω_Λ and for matter, Ω_m as follows:

$$\frac{c_1}{c_2} \approx 6 \frac{\Omega_\Lambda}{\Omega_m} \quad (4.15)$$

For $n > 0$, it appears that this model does not have an explicit cosmological constant, however for small values of Eq. (4.15), i.e., $(\frac{c_1}{c_2}) \rightarrow 0$, it behaves like cosmological constant for both local and cosmological scales [49].

4.1.3.3 Viability conditions for $f(R)$ models

There are conditions that $f(R)$ models should satisfy in order to mimic and not to deviate from Λ CDM.

The following are conditions for the viable models of $f(R)$ to hold.

- $f_{RR} > 0$, for classically stable high-curvature regime and the existence of a matter-dominated phase [5].
- $1 + f_R > 0$, this ensures that Newton's constant is positive all times and the graviton energy is positive as well [5].
- $f_R < 0$, this condition ensures that GR behavior is recovered at early times, it implies that f_R should be negative and monotonically growing function R in the range $-1 < f_R < 0$ [5].
- $|f_R| \ll 1$, this condition is responsible for the late-time cosmological evolution, it should resemble that of Λ CDM model [5, 39].

Refer to all the above conditions, a viable $f(R)$ model is carefully designed such that these conditions are satisfied. For example: It is important to establish the sign of f_{RR} to be positive. If its not, it means that there exist ghost-like degree of freedom. In Physics ghosts are necessary to keep gauge invariance in theories where the local fields exceed a number of physical degrees of freedom [51]. In modified gravity such as general infrared modifications of the Einstein-Hilbert action and scalar-tensor gravity there exist ghosts during modification and thus gives an unclear understanding of gravity at the solar system scale [52]. A clear disadvantage of this type of modification, is that the theory becomes higher dimensional at large distances and the infinite number of degrees of freedom introduced in this way is not reducible to the addition of an arbitrary function of curvature invariants [53, 54]. Therefore f_R and f_{RR} should be always positive to avoid this and guarantee the coupling to matter is positive and as to avoid the above mentioned issues. [55–57].

In this thesis we use these conditions to derive the second order equation for the density contrast for matter dominated Universe .

For example in Λ CDM model, if we consider only pure matter dominated Universe the κ coefficients are constants, given as $\kappa_1 = -0.5$, $\kappa_2 = 0.5$ and $\kappa_3 = -0.75$, where the constants are

calculated by using

$$\kappa_i \equiv \frac{\mathcal{H}^{(i)}}{\mathcal{H}^{(i+1)}}, \quad (4.16)$$

for $i = 1, 2, 3$.

For consistency, we showed that for both Λ CDM model and $f(R)$ models, the initial values for these constants are the same or should be the same based on Eq. (4.16), i.e., for $i = 1$.

In Λ CDM, for $i = 1$ we then have

$$\kappa_1 = \frac{\mathcal{H}'}{\mathcal{H}^2}. \quad (4.17)$$

Now we need \mathcal{H}' but we know that

$$\mathcal{H}' = \frac{d\mathcal{H}}{d\eta} \quad \text{and} \quad \mathcal{H} = aH \quad \text{where} \quad H = \sqrt{\frac{8\pi G}{3} \frac{\rho_{m0}}{a^3} + \frac{\Lambda}{3}} \quad (4.18)$$

After some algebra, we easily find that

$$\kappa_{1\Lambda\text{CDM}} = -\frac{1}{2} \left[\frac{\left(\frac{8\pi G}{3} \frac{\rho_{m0}}{a^3} \right)}{\left(\frac{8\pi G}{3} \frac{\rho_{m0}}{a^3} \right)} \right] \quad (4.19)$$

which is indeed $\kappa_{1\Lambda\text{CDM}} = -0.5$ as expected.

In $f(R)$ models, we do similar calculations and the results that we obtained are

$$\kappa_{1f(R)} = 1 - \left[\frac{(1+z)\hat{h}(z)_{f(R)}}{h(z)_{f(R)}} \right]. \quad (4.20)$$

where the hat represents a derivative with respect to redshift, i.e., $\hat{} \equiv \frac{d}{dz}$. The redshift z is the related to scale factor as Eq. (2.17) and $h(z)$ is the normalized Hubble parameter as a function of redshift calculated by using the Λ CDM background with the initial redshift value of $z_{in} = 10$.

After substitution, we find that the initial value of $\kappa_{1f(R)} = -0.5$ which is equal to that of Λ CDM and this makes sense because we know that at large values of redshift, the $f(R)$ should agree with Λ CDM, in our case the $f(R)$ used was that of the Hu-Sawick model. More details in the following sections.

Similarly for κ_2 the same can be done and the values for both cases were found to be $\kappa_{2\Lambda\text{CDM}} = \kappa_{2f(R)} = 0.5$.

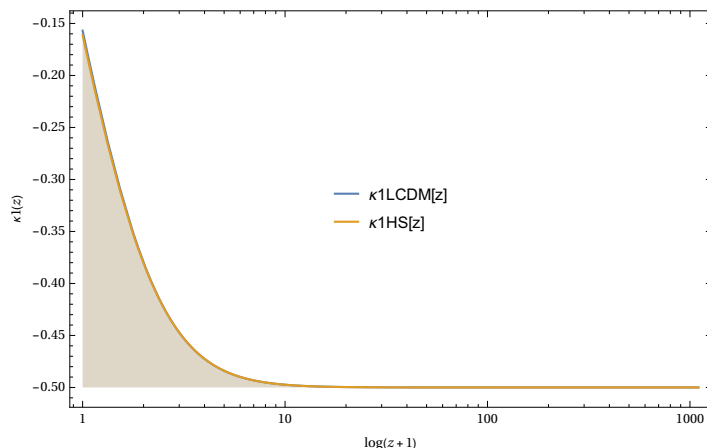


Figure 4.1: The comparison between the evolution of the κ_1 constants from both Λ CDM and $f(R)$ sharing the same initial value.

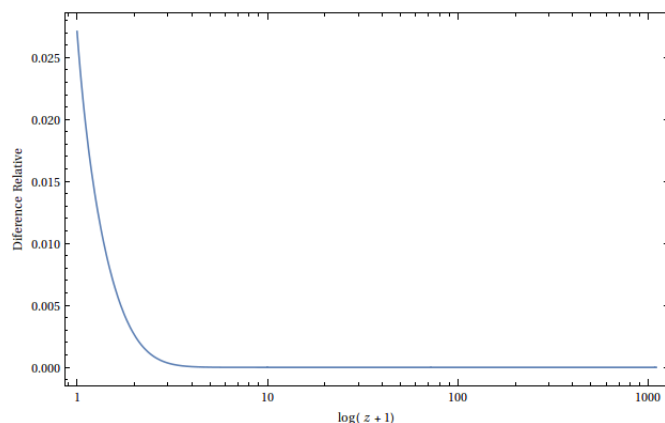


Figure 4.2: Relative difference of κ_1 both from Λ CDM and Hu-Sawicki model.

Similarly for κ_2 's for both models as well, we find that the evolution of the κ_2 's is indistinguishable for both Λ CDM and Hu-Sawicki models, except at the very low values of redshift where we see divergence takes place as depicted in Fig.(4.3).

4.1.4 Dynamical system approach to $f(R)$ (Hu-Sawicki model)

In physics, a dynamical system is described as a "particle or ensemble of particles whose state varies over time and thus obeys differential equations involving time derivatives. In order to make a prediction about the system's future behaviour, an analytical solution of such equations or their integration over time through computer simulation is realised [58]. We apply this method to solve the closed systems in cosmology particularly in $f(R)$ theories, because our equations are complicated such that there is no direct method to use.

An example in case of $f(R)$ models we have these equations Eq. (4.25) and Eq. (4.26) to solve

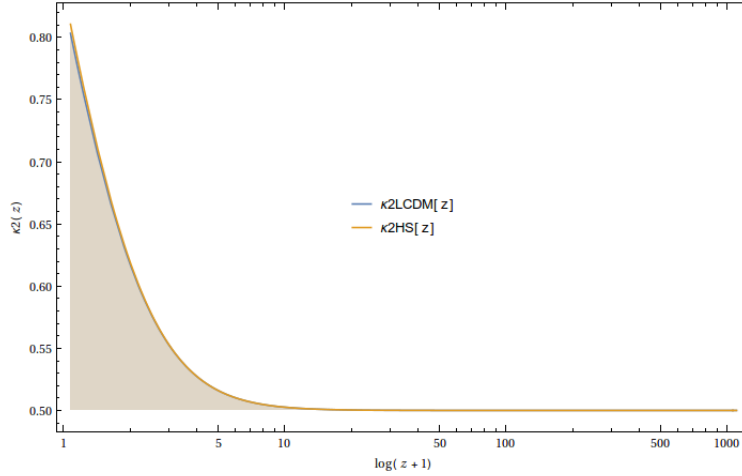


Figure 4.3: The comparison between the evolution of the κ_2 constants from both Λ CDM and $f(R)$ sharing the same initial value.

which are higher order differential equations unlike in pure General Relativity, therefore we apply dynamical system approach to solve, starting by extracting or normalizing these equations to obtain the dynamical variables which can be solved by this method.

There are other approaches, which are very useful and interesting to study perturbation theory such as the Covariant and gauge-invariant approach [59].

4.2 Cosmological perturbations in $f(R)$ theories of gravity

4.2.1 Introduction

The idea of perturbation theory expands also in the extended theories of gravity. For the purpose of this study we will be specific and focus only in one class of modified (extended) gravity theories ,i.e., the $f(R)$ theories. This section discuss perturbations in $f(R)$ theories, this include its evolution, how it grows with respect to time etc. The previous studies have shown that $f(R)$ theories, gives scientific insight about the matter density of our Universe, since they resembles one of the best known model of dark energy (Λ CDM) and they mimic its expansion history in particular. It is well understood that the evolution of perturbations depends on the specific gravity model [1] and this observation is one important aspect to differentiate between different models that can explain the cosmic acceleration of the Universe . Unlike in GR, $f(R)$ theories give a fourth-order differential equation for matter density contrast under the Longitudinal Gauge construction [5].

4.2.2 Perturbed Einstein's equations in $f(R)$ gravity theories

The Einstein's field equations can be perturbed in $f(R)$ gravity theories, in this section, we study how the equation of density perturbation in $f(R)$ evolved.

We start with the perturbed metric, in spherical coordinates

$$ds^2 = a^2(\eta) \left[- (1 + 2\Psi) d\eta^2 + (1 - 2\Phi) (dr^2 + r^2 d\Omega^2) \right]. \quad (4.21)$$

where Ψ and Φ are scalar perturbations and both are functions of space and conformal time.

Combining with the perturbed energy-momentum tensor, gives

$$\begin{aligned} \delta T_{00} &= \delta \rho = \rho_0 \delta, \\ \delta T_{ij} &= -\delta P \delta_{ij} = -c_s^2 \delta_{ij} \rho_0 \delta, \\ \delta T_{i0} &= -\delta T_{0i} = -(1 + c_s^2) \rho_0 \partial_i v. \end{aligned} \quad (4.22)$$

where ρ_0 is defined as the unperturbed energy density and v is the potential for the velocity perturbations. If we assume that our background equations hold, then the first-order perturbed equations in $f(R)$ theories are given by [5]

$$\begin{aligned} (1 + f_R) \delta G_v^\mu + (R_{0v}^\mu + \nabla^\mu \nabla_v - \delta_v^\mu \square) f_{RR} \delta R + [\nabla_v \nabla_\alpha (\delta g^{\mu\alpha}) - \delta_v^\mu (\delta g^{\alpha\beta}) \nabla_\alpha \nabla_\beta] f_R - [g_0^{\alpha\mu} \delta \Gamma_{\alpha v}^\gamma \\ - \delta_v^\mu g_0^{\alpha\beta} \delta \Gamma_{\beta\alpha}^\gamma] \partial_\gamma f_R = -\kappa \delta T_v^\mu \end{aligned} \quad (4.23)$$

where R_{0v}^μ represents the Ricci tensor that describe the components corresponding to the unperturbed FLRW metric and this is the fourth-order differential equation, unlike a second order differential equation that is obtained from standard General Relativity (Hilbert-Einstein Action). We want to compute the perturbed covariant derivatives for energy-momentum tensor T_v^μ with respect to the perturbed metric, we then obtain

$$\tilde{\nabla}_\mu \tilde{T}_v^\mu = 0. \quad (4.24)$$

In Fourier space, the calculated non-zero components ,i.e., $((00), (ii), (0i) = (i0)$ and (ij) for $i \neq j$ and $i, j = 1, 2, 3$) for the linearized Einstein's equations are obtained respectively as follows [5].

For (00) component we find

$$(1+f_R)[-k^2\Psi - k^2\Phi - 3\mathcal{H}(\Phi' + \Psi')] + (3\mathcal{H}' - 6\mathcal{H}^2)\Phi - 3\mathcal{H}'\Psi + f'_R(3\mathcal{H}\Psi - 9\mathcal{H}\Phi - 3\Psi') = 2\delta\tilde{\rho}. \quad (4.25)$$

Similarly for (ii) component we find

$$(1+f_R)[\Psi'' + \Phi'' + 3\mathcal{H}(\Phi' + \Psi') + 3\mathcal{H}'\Phi + (\mathcal{H}' + 2\mathcal{H}^2)\Psi] + f'_R(3\mathcal{H}\Phi - 9\mathcal{H}\Psi - 3\Phi') + f''_R(3\Phi - \Psi) = 2\delta c_s^2\tilde{\rho}, \quad (4.26)$$

and for $0i = i0$ we obtain

$$(1+f_R)[\Psi' + \Phi' + \mathcal{H}(\Phi + \Psi)] + f'_R(2\Phi - \Psi) = -2v(c_s^2 + 1)\tilde{\rho}. \quad (4.27)$$

Finally the (ij) component is obtained when $i \neq j$ for $i, j = 1, 2, 3$

$$(\Phi - \Psi)(1+f_R) = -f_{RR}\delta R \quad (4.28)$$

where δR is the change in the Ricci scalar between the perturbed and the unperturbed ,i.e., $\delta R = (R - R_0)$ is defined as

$$\delta R = -\frac{2}{a^2}[3\Psi'' + 6(\mathcal{H}' + \mathcal{H})\Phi + 3\mathcal{H}(\Phi' + 3\Psi') - k^2(\Phi - 2\Psi)]. \quad (4.29)$$

We note that the Energy-momentum tensor Eq. (4.24) does not depend on $f(R)$, therefore the first-order equations read as follows:

$$3\Psi'(1+c_s^2) - \delta' + k^2(1+c_s^2)v = 0, \quad (4.30)$$

and

$$\Phi + \frac{c_s^2}{1+c_s^2}\delta + v' + \mathcal{H}v(1-3c_s^2) = 0. \quad (4.31)$$

In matter dominated Universe ,i.e., for $c_s^2 = 0$, we can use Eq. (4.30) and Eq. (4.31) to obtain

$$\delta'' + \mathcal{H}\delta' + k^2\Phi - 3\Psi'' - 3\mathcal{H}\Psi' = 0, \quad (4.32)$$

which is the evolution equation for density perturbations in $f(R)$ gravity and for matter dominated Universe ,i.e., $c_s^2 = 0$ the Ψ and Φ are given as follows [5]:

$$\begin{aligned} \Phi = \frac{1}{D(\mathcal{H}, k)} & \left[\left(3(1+f_R)\mathcal{H}(\Psi' + \Phi') + f'_R\Psi' + 2\tilde{\rho}\delta \right) (1+f_R)(\mathcal{H} - f'_R) + \left((1+f_R)(\Phi' + \Psi') + \frac{2\tilde{\rho}}{k^2}(\delta' - 3\Psi') \right) \right. \\ & \left. + \left((1+f_R)(-k^2 - 3\mathcal{H}') + 3f'_R\mathcal{H} \right) \right], \end{aligned} \quad (4.33)$$

and

$$\Psi = \frac{1}{D(\mathcal{H}, k)} \left[\left(-3(1+f_R)\mathcal{H}(\Psi' + \Phi') - 3f'_R\Psi' - 2\tilde{\rho}\delta \right) \left((1+f_R)\mathcal{H} + 2f'_R \right) - \left((1+f_R)(\Phi' + \Psi') + \frac{2\tilde{\rho}}{k^2}(\delta' - 3\Psi') \right) + \left((1+f_R)(-k^2 + 3\mathcal{H}' - 6\mathcal{H}^2 - 9\mathcal{H}f'_R) \right) \right], \quad (4.34)$$

where

$$D(\mathcal{H}, k) \equiv -6(1+f_R)^2\mathcal{H}^3 + 3\mathcal{H} + 3\mathcal{H}(f_R'^2 + 2(1+f_R)^2\mathcal{H}') + (1+f_R)f'_R(-2\mathcal{H}^2 + k^2 + \mathcal{H}'). \quad (4.35)$$

Finally the Anisotropic stress equation in these theories is given by [60],

$$\Pi_{f(R)} = \left[f_{RR}\delta R + (1+f_R)(\Phi - \Psi) \right]. \quad (4.36)$$

4.2.2.1 Modified Friedmann Equations

For a spatially flat Universe, we consider a Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, which in Cartesian coordinates can be written as:

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 \quad (4.37)$$

By using Eq. (4.7) and Eq. (4.28), we derive the non-trivial field equations (**Modified Friedmann and Raychaudhuri**) [45]:

$$3H^2 = \frac{1}{f_R} \left(\rho_m + \rho_r + \frac{Rf_R - f}{2} - 3Hf_{RR}\dot{R} \right), \quad (4.38)$$

$$3\dot{H} + 3H^2 = -\frac{1}{2f_R} \left(\rho_m + 3p_m + f - f_R R + 3Hf_{RR}\dot{R} + 3f_{RRR}\dot{R}^2 + 3f_{RR}\ddot{R} \right), \quad (4.39)$$

where ρ_m and ρ_{rad} represent the matter and radiation densities respectively, satisfying the continuity equations:

$$\begin{aligned} \dot{\rho}_m + 3H\rho_m &= 0 \\ \dot{\rho}_r + 4H\rho_r &= 0 \end{aligned} \quad (4.40)$$

Note: the Eq. (4.38) and Eq. (4.39) are written in-terms of cosmic time corresponding to Eq. (4.8) and Eq. (4.9), respectively.

4.2.2.2 Modified Newtonian constant

One of the interesting features of the extended theories of gravity is the fact that the universal gravitational constant is time and scale dependent [61].

Let us add a small scalar perturbation to the metric (in the Newtonian gauge) in order to investigate this.

Consider Eq. (3.29), where Φ and Ψ are the Newtonian potentials and use the field equations to first order. After some algebraic manipulation, one can define a Poisson equation in the Fourier space and attribute the extra terms that appear on the right-hand side to an effective gravitational constant G_{eff} [61]. When considering sub-horizon scales ,i.e., $k^2 \gg a^2 H^2$, we obtain the gravitational potential:

$$\Phi = -4\pi G_{\text{eff}} \left(\frac{a^2 \delta \rho_m}{k^2} \right), \quad (4.41)$$

where $\delta \rho_m$ is a perturbation in the matter density, k is the Fourier scale and

$$G_{\text{eff}} = \frac{1}{8\pi f_R} \left[\frac{1 + 4 \frac{k^2}{a^2 R} m}{1 + 3 \frac{k^2}{a^2 R} m} \right]. \quad (4.42)$$

Here m is related to the Ricci scalar as [61]

$$m \equiv \frac{R f_{RR}}{f_R}. \quad (4.43)$$

4.2.3 Summary for Chapter 4

Chapter 4 discussed deeply the question " Why Modify Gravity ? ", addressing the issues that are facing our standard model of cosmology and giving tips and ideas about how one can go about in an attempt to solve these issues, hence the strong motivation for Modifying General Relativity and use it in doing so. The background $f(R)$ equations are briefly discussed and fully explained in Chapter 5 for some reasons. The importance of cosmological perturbation theory in $f(R)$ theories of gravity was introduced as well. We discussed the field equations under the class of $f(R)$ and showed how to obtain the set of Modified Einstein's field equations, the Modified Friedmann equations as well as the Modified Newtonian constant. Deep analysis of Modified Friedmann equation appears in the Chapter 5 where it is applied to Hu-Sawicki model.

EVOLUTION OF DENSITY PERTURBATIONS IN $f(R)$ THEORIES OF GRAVITY

5.1 Evolution of density perturbations in $f(R)$ theories of gravity

5.1.1 Introduction

At this point, we should recall that the cosmological principle is valid only on large scales but not on scales smaller than the size of the horizon. The idea is that the Universe started off in an extremely homogeneous and isotropic state. Inflation provided with the initial conditions as the accelerated phase of expansion started to occur and the Universe started to cool off. As the Universe starts to expand, the tiny primordial density perturbations generated from quantum fluctuations of the vacuum and these density fluctuations would grow under the influence of gravity and eventually collapse to form the structures that we observe today, i.e., galaxies, clusters, etc.

5.1.2 Evolution of matter density perturbations in Λ CDM

In order to investigate how matter density perturbations evolve in $f(R)$ theories, let us first discuss the growth of density perturbations in Λ CDM model. By now we should know that Λ CDM model considers "dark energy fluid", which is described by the " Λ " term in the Einstein's equations, given by Eq. (1.25).

When we consider metric formalism in Λ CDM model, we can easily obtain the growth of matter

density perturbations of second order form of differential equation $\delta \equiv \frac{\delta\rho}{\rho_0}$ [5]. where

$$\delta = \frac{\rho(\vec{x}, t) - \rho_0(t)}{\rho_0(t)}.$$

Recall the Einstein tensor Eq. (2.1).

A flat Friedmann-Lemaître-Robertson-Walker metric is considered to study scalar perturbations in the Newtonian gauge is described by Eq. (4.24).

The field equations for the perturbations is

$$\delta G_{\mu\nu} = -8\pi G \delta T_{\mu\nu} \quad (5.1)$$

Note, we consider the natural units hence the speed of light $c = 1$ and the corresponding perturbed energy-momentum tensor obtained is given by Eq. (4.25) [5].

At this point, it is reasonable enough to assume that at the background level the perturbed and unperturbed matter density have the same evolution equation of state.

$$\frac{\delta P}{\delta\rho} \equiv c_s^2 \equiv \frac{\delta P_0}{\delta\rho_0}. \quad (5.2)$$

For matter perturbations, $c_s^2 = 0$. The following is differential equation for δ in Fourier space [5]

$$\delta'' + \mathcal{H} \frac{k^4 - 6\tilde{\rho}k^2 - 18\tilde{\rho}^2}{k^4 - \tilde{\rho}(3k^2 + 9\mathcal{H}^2)} \delta' - \tilde{\rho} \frac{k^4 + 9\tilde{\rho}(2\tilde{\rho} - 3\mathcal{H}^2) - k^2(9\tilde{\rho} - 3\mathcal{H}^2)}{k^4 - \tilde{\rho}(3k^2 + 9\mathcal{H}^2)} \delta = 0, \quad (5.3)$$

where

$$\tilde{\rho} \equiv 4\pi G \rho_0 a^2(t), \quad \mathcal{H} \equiv \frac{a'(t)}{a(t)}.$$

• $a(t)$ is a cosmic scale factor and $'$ represent a derivative with respect to conformal time η and the cosmic scale is related to the redshift as given by Eq. (2.16),

5.1.3 Evolution under Sub-Horizon approximation

At this point, the interest is to study how matter density perturbation behave under sub-horizon approximation, i.e., $k \gg \mathcal{H}$. In this approximation, Eq. (4.26) reduces to, [5].

$$\delta'' + \mathcal{H}\delta' - 4\pi G \rho_0 a^2 \delta = 0. \quad (5.4)$$

It has been studied that, at early times in the sub-Hubble regime the matter energy density dominates over the cosmological constant and it grows as the cosmic scale factor,

$$\delta(a) \propto a(\eta). \quad (5.5)$$

If we consider late-times, i.e., today, we cannot ignore the contribution from the cosmological constant, therefore the power-law solutions of Eq. (5.4) no longer exist. Hence the following form of solution for Eq. (5.4) was proposed [5]:

$$\frac{\delta(a)}{a} = e^{\int_{a_i}^a (\Omega_m(a)^\gamma - 1) d \ln a}. \quad (5.6)$$

Eq.(5.6) fits well with the known numerical solution for δ with a constant $\gamma = 6/11$ and for that reason, it has been considered to be a success.

It is reasonable to investigate how the growth of matter density perturbations behaves in sub-Hubble limit in the $f(R)$ point of view. In this limit, i.e., $\mathcal{H}/k \ll 1$ the evolution perturbation equation is given by [5] and the background quantities κ_i are discussed in Section(4.1.3.3):

$$\delta'' + \mathcal{H}\delta' + \frac{(1+f_R)^5 \mathcal{H}^2 (-1+\kappa_1)(2\kappa_1-\kappa_2) - \frac{16}{a^8} f_{RR}^4 (\kappa_2-2) k^8 8\pi G \rho_0 a^2}{(1+f_R)^5 (-1+\kappa_1) + \frac{24}{a^8} f_{RR}^4 (1+f_R)(\kappa_2-2) k^8} \delta = 0 \quad (5.7)$$

In the literature such as [62], it has been shown that when one performs some theoretical calculations by using the perturbed equations Eq. (4.19), Eq. (4.22), Eq. (4.24) and Eq. (4.25) and neglecting the time derivatives of Ψ and Φ in the process, the previous equation reduces to

$$\delta'' + \mathcal{H}\delta' - \left[\frac{\frac{f_{RR}}{(1+f_R)} \frac{4k^2}{a^2} + 1}{\frac{f_{RR}}{(1+f_R)} \frac{3k^2}{a^2} + 1} \right] \left(\frac{\tilde{\rho}}{1+f_R} \right) \delta = 0 \quad (5.8)$$

Eq.(5.8) has been considered to be too aggressive [5] since the evolution equation is time-dependent, meaning that removing time-dependent terms might remove some important information about the evolution itself [63].

For viable models of $f(R)$ in the sub-horizon limit, we obtain the differential equation of this nature:

$$\delta'' + \mathcal{H}\delta' - \frac{4}{3} \left[\frac{\left(\frac{6f_{RR}k^2}{a^2} + \frac{9}{4} \right)^2 - \frac{81}{16} + \frac{9}{2} \left(\frac{2\kappa_1 - \kappa_2}{\kappa_2 - 2} \right)}{\left(\frac{6f_{RR}k^2}{a^2} + \frac{5}{2} \right)^2 - \frac{25}{4} + \frac{12}{2} \left(\frac{\kappa_1 - 1}{\kappa_2 - 2} \right)} \right] (1 - \kappa_1) \mathcal{H}^2 \delta = 0. \quad (5.9)$$

Since a viable $f(R)$ should satisfy $|f_R| \ll 1$, in this limit it can be proven that $\kappa_1 - \kappa_2 = 0$, therefore $2\kappa_1 - \kappa_2 \approx -2 + \kappa_2 \approx -1 + \kappa_1$, which allows simplifying expression Eq. (5.9) to approximately become Eq. (5.8) and κ_i is given by Eq. (4.17).

The evolution equations (5.3), (5.7) and (5.9) will be later solved by using the background values for both Λ CDM model and the $f(R)$ model specifically the Hu-Sawicki model in our case, to test the possible deviations between the two models.

5.2 Numerical solution of the Evolution Equations

5.2.1 Solving the Hu-Sawicki model using dynamical system method in the FLRW Universe

Dimensionless dynamical variables

To achieve our goal, we start with our modified Friedmann and Raychaudhuri equations, i.e., Eq. (4.32) and Eq. (4.33). We define the following dimensionless dynamical variables [49, 64]:

$$x \equiv \frac{\dot{R}f_{RR}}{Hf_R}, \quad y \equiv \frac{R}{6H^2}, \quad \chi \equiv \frac{f}{6H^2f_R}, \quad \tilde{\Omega}_m \equiv \frac{\mu_m}{3H^2f_R}, \quad h(z) \equiv \frac{H}{H_0}. \quad (5.10)$$

Please note, we are using the idea of spatially flat Universe, with our FLRW metric to do calculations. At this stage, we take redshift z derivatives of these dimensionless dynamical variables $(x, y, \chi, \tilde{\Omega}_m, h)$ in Eq. (5.7), which leads to a system of first order equations obtained as [62]:

$$\begin{aligned} \frac{dh}{dz} &= \frac{1}{(1+z)} \left[(2-y)h \right], \\ \frac{dx}{dz} &= \frac{1}{(1+z)} \left[x^2 - x(y+1) - 2y + 4\chi - \Omega \right], \\ \frac{dy}{dz} &= \frac{1}{(1+z)} \left[y(2y - xQ - 4) \right], \\ \frac{d\chi}{dz} &= \frac{1}{(1+z)} \left[\chi(x + 2y - 4) - xyQ \right], \\ \frac{d\tilde{\Omega}_m}{dz} &= \frac{1}{(1+z)} \left[\Omega(x + 2y - 1) \right], \end{aligned} \quad (5.11)$$

where $Q = \frac{f_R}{Rf_{RR}}$.

In order for us to solve the above, we need to close the system and to do so, we rewrite the term Q in terms of dynamical variables [62], which is found to be

$$Q = \frac{(7 + 10h^2y)^3 \left[1 - \frac{49}{(7+10h^2y)} \right]}{980h^2y}. \quad (5.12)$$

We also need to fix the initial conditions for the normalized Hubble parameter $h(z)$ and the decelerating parameter $q(z)$ as well as our average density Ω_0 fixed today. The constraint Friedmann equation in terms of dynamical system variables, is given by

$$\tilde{\Omega}_m + y - \chi - x = 1. \quad (5.13)$$

Therefore we can compute the initial conditions for $(y, \chi$ and $\tilde{\Omega})$ and then after we obtain the initial value for x straight from the constraint equation Eq. (5.10) [64].

Initial conditions

Let us start by defining our normalized Hubble parameter as

$$h(z) = \frac{H}{H_0} = \sqrt{\Omega_{mo}(1+z)^3 + \Omega_\Lambda}, \quad (5.14)$$

with the values of $\Omega_{mo} = 0.3, \Omega_\Lambda = 0.7$ and the redshift $z_{in} = 10$.

Similarly the initial value of y , is given by

$$y_{in} = \frac{R(z_{in})}{6h(z_{in})^2}. \quad (5.15)$$

For χ_{in}

$$\chi_{in} = \frac{f(R_{in})/H(z_{in})^2}{6f_R h(z_{in})^2}, \quad (5.16)$$

where the initial value of the first derivative of the $f(R)$ is calculated as $f_R = 0.999987$.

Finally

$$\tilde{\Omega}_m = \frac{\mu_m/H_0^2}{3f_R h(z_{in})^2}. \quad (5.17)$$

The initial values obtained for the dynamical system variables are presented in Table 5.1.

Initial values for dynamical variables	
z_{in}	10
$h(z_{in})$	20
$\tilde{\Omega}_m$	0.9983
χ_{in}	-0.500866
y_{in}	0.50300
x_{in}	-0.0073

Table 5.1: obtained dynamical variables initial values.

The studied values of n that appear in Eq. (4.17) vary in the interval $[1, 2]$.

NB: The values of n can be any number greater than zero but for the purpose of this work, we only check in the interval given above.

Decelerating parameter

In FLRW Universe, the cosmic acceleration expansion of space is measured by using the cosmological dimensionless parameter q [65].

This parameter in Λ CDM model it is given by

$$q_{\Lambda CDM} = 1 - \frac{\Omega_m(z+1)^3 + 4\Omega_\Lambda}{2\Omega_m(z+1)^3 + 2\Omega_\Lambda}, \quad (5.18)$$

whereas for our model, i.e., Hu-Sawicki model, we obtain the deceleration parameter in terms of dynamical variables as

$$q_{HS} = 1 - y. \quad (5.19)$$

It is logically reasonable to compare the two equations Eq. (5.15) and Eq. (5.16) to see the possible deviations. By using the above initial conditions the comparison was done and the results obtained are presented below. To be specific, we find that the deceleration parameter is enhanced in our modified gravity theory with respect to Λ CDM and $n = 2$, $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $z_{in} = 6$ and $y_{in} = 0.503$, see Fig. (5.1).

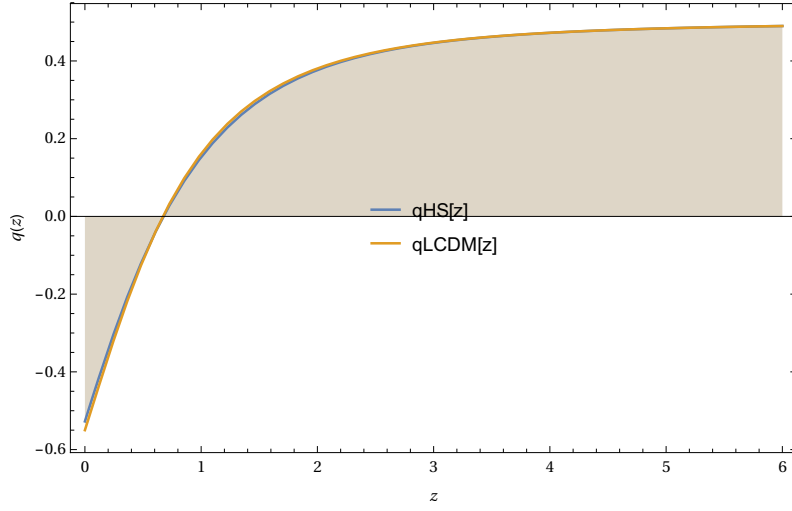


Figure 5.1: Decelerating parameter q vs redshift z , at high redshifts values, the two models coincide for a while and start to deviate at low redshifts around $z = 2$ and then meet again at about $z = 0.7$ and split again at about $z = 0.4$.

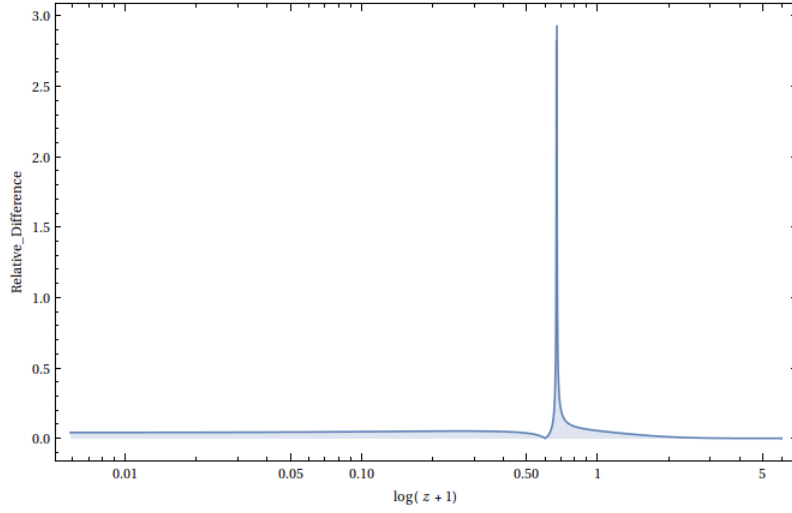


Figure 5.2: This shows the relative difference between the two models

The evolution of the normalised Hubble parameter $h(z)$ was also studied, its behaviour against the redshift to test the expansion of the Universe, as we can see in Fig. (5.3), both models grow the same and they match significantly with an error of about 0.5 percent, which appears at low redshift values.

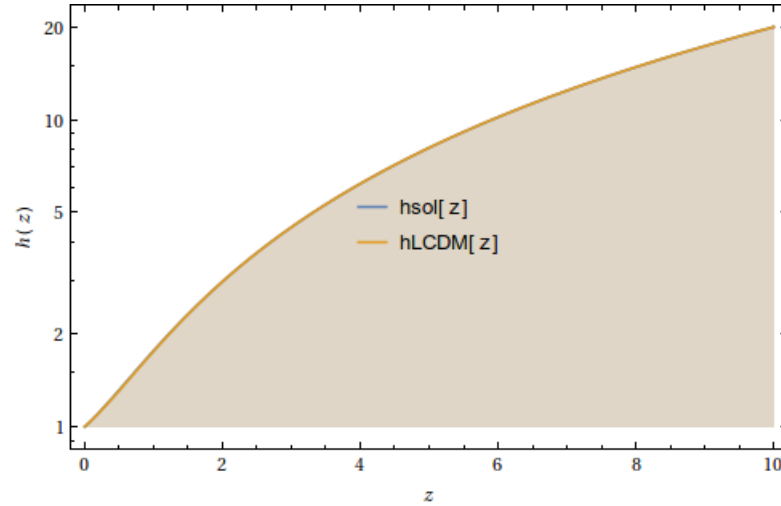


Figure 5.3: Normalised Hubble parameter h vs. redshift z , at all redshifts values, the two models coincide and gives the expected results.

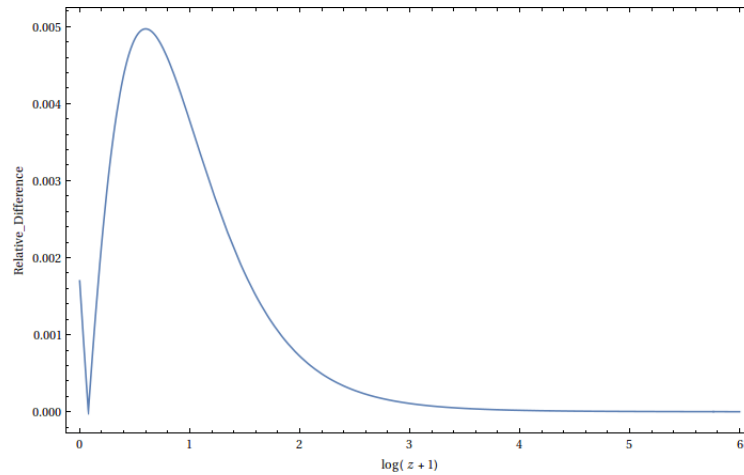


Figure 5.4: This shows the relative difference for the normalized Hubble parameter between the two models.

Testing the constraint equation for $n = 2$

It is also important to check the consistency of the constraint equation (5.13). The results proved that, indeed the initial values for dynamical system variables were found to be true.

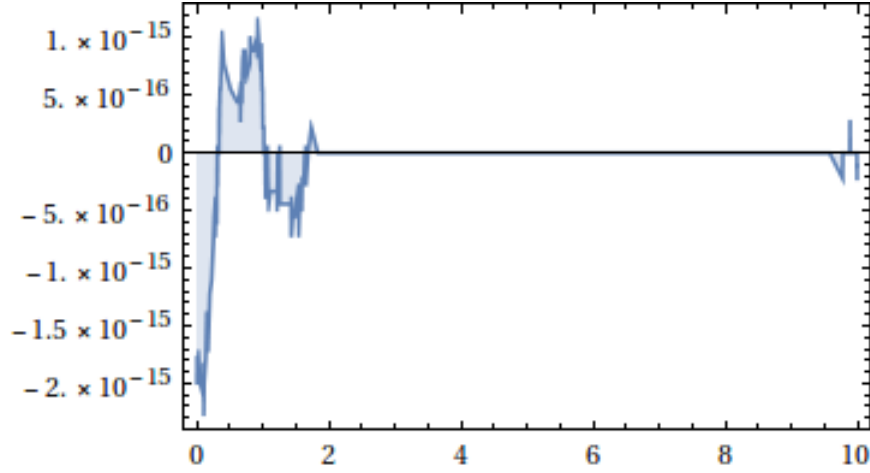


Figure 5.5: The constraint equation vs. redshift, to test theoretical accuracy, indeed it gives the expected results. It is constant at zero almost everywhere, at $z = 0$ to about $z = 2$ we see an unexpected behaviour although this can be neglected since it is very small (very close to zero).

The following table presents the values of $h(z)$ and $q(z)$ calculated today, i.e., where $H = H_0$ and $\Omega_0 = 0.3$ for different values of exponent n .

The Hu-Sawicki model provide h_0 and q_0 values which give the values of H and q that match with those known from Λ CDM model and the data was obtained by varying the value of n and fixing the z redshift value.

n values	q_0	h_0
1	-0.2274	0.9405
1.1	-0.2986	0.9655
1.4	-0.4224	0.9918
1.8	-0.5051	0.9967
2	-0.5274	0.9983
Λ CDM	-0.55	1.00

Table 5.2: values of q_0 and q_0 obtained from the Hu-Sawicki model by varying the value of n while fixing the z redshift value

The Λ CDM value for q_0 was obtained by using Eq. (5.16) and we see that for the value of redshift and for $n = 2$, both models give almost the same values of q_0 and h_0 .

5.2.2 Equivalence of theories

Now that we have studied the theory of both Λ CDM and Hu-Sawicki models, it is reasonable and important enough to do all possible tests and compare the results that we are getting to the predictions of the theory. Because of the level of difficulties and order of degrees of freedom, the density perturbation equations are unable to solve analytically, as a result we use numerical method in order to solve. This section presents the results obtained after calculations. We previously mentioned that, when varying the Hilbert-Einstein's Action+ Λ with respect to the metric and a differential equation for density perturbation is obtained as Eq. (5.3) and for $k \gg \mathcal{H}$ in the sub-Hubble modes, Eq. (5.3) reduces to

$$\delta'' + \mathcal{H}\delta' - \tilde{\rho}\delta = 0. \quad (5.20)$$

When we use background values of Λ CDM model and the background values from the Hu-Sawicki model and feed both of those in the equation Eq. (5.3) and plot them against redshift, the following results were obtained. This calculation was done by considering the redshift range from 1100 to 0 and $n = 2$, $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$. and $k = 100H_0$.

Note: we chose $n = 2$, otherwise we could have chosen any value to analyse in our interval, refer to Table (5.1).

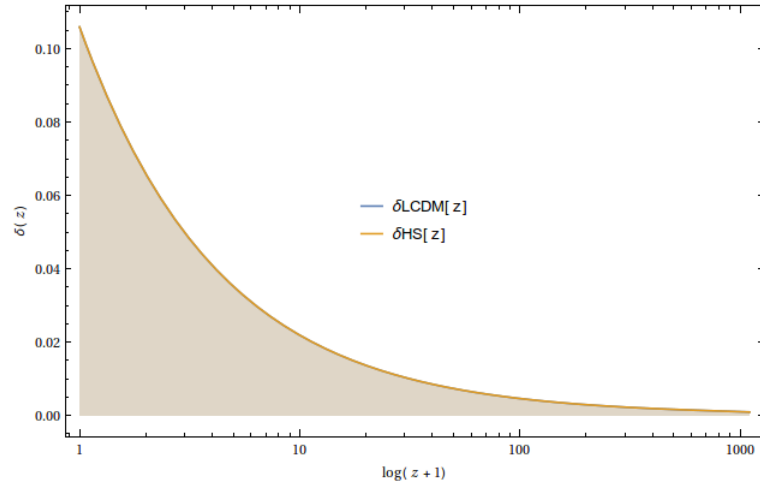


Figure 5.6: The evolution of density perturbations is indistinguishable for both Λ CDM and the Hu-Sawicki model

We did the same calculation but using the Quasi-static equation Eq. (5.8), and investigate the differences between the two models. After using our cosmological background for both models and the redshift range from 1100 to 0 and $n = 2$, ($\delta_{HS} = \delta_{\Lambda CDM} = 9.082 \times 10^{-4}$), Similarly ($\delta'_{HS} = \delta'_{\Lambda CDM} = -8.249 \times 10^{-7}$) and the results obtained are presented in Fig (5.5).

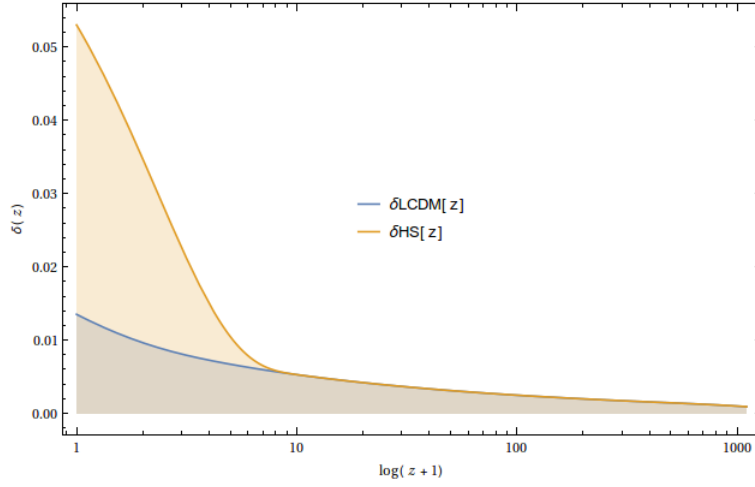


Figure 5.7: The quasi-static evolution is indistinguishable at high values of redshift but diverges from Λ CDM as the redshift decreases and fails to resemble Λ CDM as expected.

We also used Eq. (5.3) plotted against sub-Hubble approximation given by Eq. (5.9) as well and we obtained the following, similarly the redshift range from 1100 to 0 and $n = 2$, $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$. and $k = 100H_0$.

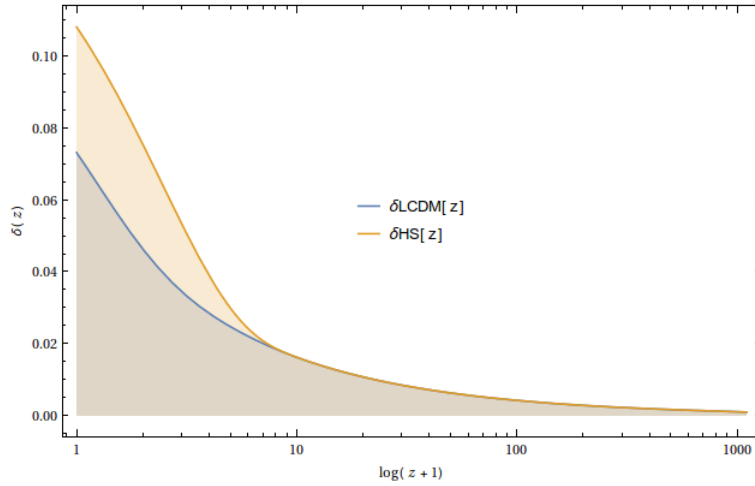


Figure 5.8: The approximation is indistinguishable at high values of redshift but diverges from Λ CDM as the redshift decreases, but it gives a good resemble to that of Λ CDM.

Here all the three equations were plotted together, one Λ CDM Eq. (5.3) and the other two are the approximations Eq. (5.8) and Eq. (5.9).

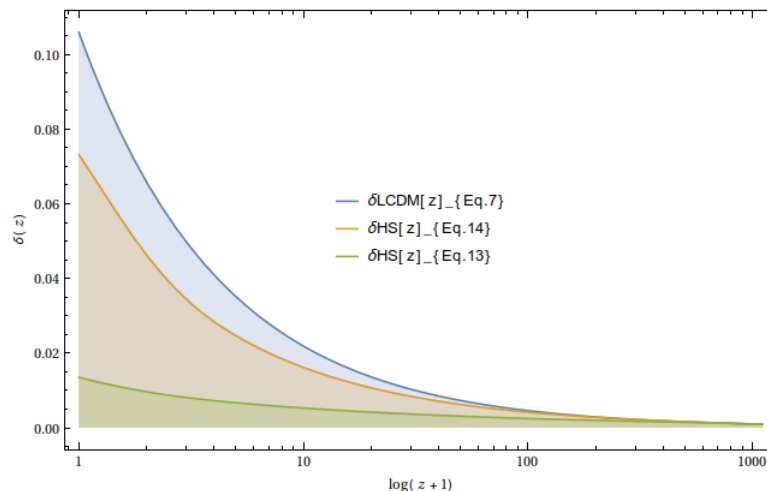


Figure 5.9: This shows how both approximations Eq. (5.8) and Eq. (5.9) diverge from Λ CDM Eq. (5.3) as the redshift decreases, plotted in the same range

5.2.3 The growth factor in the matter-dominated Universe

The perturbed conservation and acceleration equations lead to the Newtonian perturbation equations for a flat matter-dominated Universe:

$$\dot{\delta}_m + 3H\delta_H = 0, \quad H\dot{\delta}_H + (\dot{H} + 2H^2)\delta_H = \frac{4\pi G}{3}\bar{\rho}_m\delta_m. \quad (5.21)$$

where δ_H describes the density perturbation in the Hu-Sawicki model. The above equations holds in the Newtonian approximation for a flat Λ CDM model, when the radiation is neglected then it leads to

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0, \quad (5.22)$$

where δ_m is related to growth rate as [66],

$$f_{\Lambda\text{CDM}} = \frac{d \ln \delta_m}{d \ln a}. \quad (5.23)$$

The idea is to check how the growth factor in Λ CDM model is compared to the Hu-Sawicki model. The numerical results for both models were obtained as depicted in Fig. 5.10.

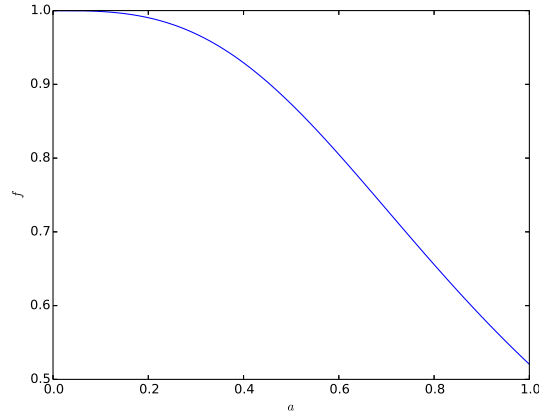


Figure 5.10: Shows a behavior of growth function in Λ CDM against the cosmic scale factor,i.e., (f vs a).

In Hu-Sawicki model, we find that the growth factor is given by

$$f_{HS} = -(1+z) \frac{\delta(z)'_{HS}}{\delta(z)_{HS}}. \quad (5.24)$$

The following are the results obtained for the growth factor using the Hu-Sawicki background values.

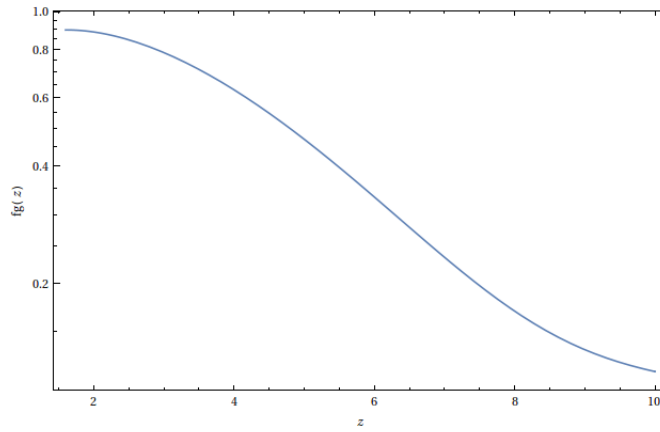


Figure 5.11: shows a behavior of growth function in $f(R)$ against the cosmological redshift,i.e., (f vs z).

where $\delta(z)'_{HS}$ and $\delta(z)_{HS}$ are taken from Eq. (5.10) after using the initial conditions from solving the dynamical system.

5.2.4 Results discussion and the comparison of the two models

We have studied/tested a class of $f(R)$ models that accelerate the expansion without a cosmological constant. We learned that its parameters allow gravitational behaviour that exhibited in cosmological, galactic and solar-system scales. Indeed the results obtained are reasonable and coincide with Λ CDM to a certain degree.

This section discusses these results. Because $f(R)$ theory is an extension of General Relativity, the best reference and ideal procedure will be to make sure that we maintain all the features of GR in all time and improve where GR needs improvement. An example in this thesis we compare the $f(R)$ results with those from Λ CDM model. From Λ CDM model the equation that describes the density behaviour of the Universe is given by Eq. (5.6), and since we are working on sub-Hubble modes, one of the most popular and most interesting results of this nature in $f(R)$ is that presented in Eq. (5.8), i.e., the Quasi-static approximation.

It has been discussed in the literature and considered that Quasi-static approximation is very aggressive and based on how it was constructed, it is suspected that some information has been or might have been lost during the process, following that it ignores some time derivatives terms. In this thesis it has been shown by taking into account all viability condition for $f(R)$ and using Eq. (4.19) the Quasi-static approximation can be improved as results we obtained Eq. (5.9).

In Fig. (5.4) we did a test and used the Λ CDM cosmological background values for both Λ CDM model and the Hu-Sawicki model and the results obtained matched, which was a success because that was expected for a viable model of $f(R)$. At this point, we know that our function of $f(R)$ is giving us a good approximation. We then did the same but using Eq. (5.8) and the Quasi-static approximation failed to mimic Λ CDM well. see Fig. 5.5. Nevertheless when we consider the $|f_R| \ll 1$ limit, we have $2\kappa_1 - \kappa_2 \approx -2 + \kappa_2 \approx -1 + \kappa_1$ and this allows us to simplify expression Eq. (5.9) to approximately become Eq. (5.8) and this is to ensure that for viable models of $f(R)$ the background evolution resembles that of Λ CDM model. Meaning that even though Eq. (5.8) failed to resemble Λ CDM model at low curvature values, for the viable $f(R)$ (i.e., Hu-Sawicki) model the Quasi-static approximation gives a correct description for the evolution of perturbations. see Fig.(5.6) and Fig. (5.7), Finally we did check the behaviour of the growth factor in $f(R)$ which was found not to be in agreement to that found in Λ CDM model, even though we did not do a thorough comparison for these but clear features are observed.

5.2.5 Summary for Chapter 5

In this chapter, We discussed mostly the numerical solution for the equations derived / studied. One was to introduce the model, i.e., Hu-Sawicki and it was clearly explained that the model was chosen among others, because of its viability and that it explains the late-phase accelerating Universe precisely as Λ CDM model which was considered to be our reference in this thesis. The evolution of perturbations via the Quasi-static approximation was also explained in details and why it is considered to be too aggressive and we introduced the method that can improve the Quasi-static approximation, since we do not want to lose its generality that, it is recovered in the matter dominated era and we want to integrate from then to today. The expansion in both models was explained via a decelerating parameter and normalized Hubble parameter, the constraint equation was also discussed as well as the obtained results are explained.

CONCLUSIONS AND OUTLOOK

6.1 Conclusions

6.1.1 Problem review

Cosmology is facing one of the biggest and the most important questions that can reveal the truth about our universe. The accelerating expansion of our universe today is one among the others. Obviously, the study shows, (at least based on Λ CDM) that this is caused by so-called "Dark Energy" but the Λ CDM model is failing to give the fundamentals of this "fluid" dark energy. The astrophysics observations, particularly the study of galaxy rotation curves implies that there exists some form of matter that is unobserved at the moment, hence called dark matter. This also points to the theory of cosmology which still does not have a solid answer. The question of structure formation as well as one of the same kind, and the study of perturbation theory is attempting to answer these types of questions, particularly the linear perturbations theory. It allows us to explore the structure formation of the universe, studying the evolution of these fluctuations (perturbations) is essential, as it helps us to understand the origins of the structure, hence our study explored the dynamics of growth density perturbations in matter dominated phase. The problem of inhomogeneity in small scales, makes our assumption or principle of cosmology not to hold. The field of cosmology is really trying hard to explain and solve these issues and many more others. In this thesis the focus was to test the other alternatives that can explain this late-time accelerating expansion phase of the universe differently from Λ CDM without losing its naturality. We also explored intensively how the growth of density perturbations in these models is and how it can be used to explain the evolution of dark sector in the universe.

6.1.2 Results overview

We have studied the evolution of matter density perturbation in $f(R)$ theories of gravity using the Hu-Sawicki model. We not only derived the equation for density perturbation in the longitudinal gauge but we also obtained the numerical solution for it, following that we could not solve it analytically. Even though we obtained a fourth-order differential equation for density perturbations under these extended theories of gravity, we have shown that in the sub-Hubble limit the equation reduces to a second-order equation which can be compared with Eq.(5.9). In these models, the evolution of matter density perturbations with Λ CDM and the Hu-Sawicki model parameters has been studied and the corresponding results have been shown in Fig.(5.5), Fig.(5.6), Fig.(5.7), respectively, which indicate that during the early times, the evolutionary trajectories of our considered model are similar to those of the Λ CDM, for both Quasi-static equation Eq.(5.13) and Eq.(5.14). And the results do agree with the theory, a complete $f(R)$ model of gravity is expected to recover GR at early-times of evolution and mimic the Λ CDM model at late-time, which indeed shown. see Fig.(5.7). Moreover, we have seen that unlike in GR, where the Bardeen potentials are equal, in modified gravity theories this is not necessarily true. And also nature of density perturbation is not the same as well. In $f(R)$ we have fourth-order differential equation as compared to the second-order that comes from normal GR, similarly the anisotropic stress as well in modified gravity it is not considered to be zero. Finally we need to point out this, the Hu-Sawicki model that we used, is constructed, as several other broken power law models, with the intention to evade the solar system tests of GR by design. Moreover, the Hu-Sawicki $f(R)$ can be considered as a natural extension to the standard Hilbert-Einstein action which is able to recover GR predictions at high curvatures and provides late-time acceleration and the density contrast. The Initial conditions must be set at high redshifts, where the behaviour is understood and integrated out to the future where we wish to investigate viability, hence in our results we see that the growth of matter on linear scales is enhanced in our modified gravity theory with respect to Λ CDM. The matter power spectra can help to disentangle the feature of the underlying correct theory of gravity. The purpose of the work presented in this thesis was to analyse the matter density perturbation of the Hu-Sawicki model, using the dynamical systems approach to cosmology, to draw out the qualitative facts provided about the universe which it governs.

6.2 Outlook

6.2.1 Future work

My plan and interest are to continue to do investigations of perturbations in general of this model also possible to consider the case and its response in a universe where flatness is not assumed. Moreover to obtain a deeper understanding of the effect of these higher-order corrections and relativistic effects in the growth of astrophysical structures initially. To clarify the validity of

several approximations (quasi-static, sub-Hubble, etc.) which are usually considered in the existing literature, despite the fact that they were proved to be very aggressive removing important information about the evolution of structures. I would like to investigate the response of the Cosmic Microwave Background (CMB) spectrum and tensor perturbations to the modification of the theory of gravity.

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